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A Construction of Gradings of Lie Algebras

Antonio Fernández López[1], **Esther García**[2], **Miguel Gómez Lozano**[1] **and Erhard Neher**[3]

[1]Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain, [2]Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain, and [3]Department of Mathematics and Statistics, University of Ottawa, Ontario K1N 6N5, Canada

Correspondence to be sent to: Esther García, Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain. e-mail: esther.garcia@urjc.es

In this article we present a method to construct gradings of Lie algebras. It requires the existence of an abelian inner ideal B of the Lie algebra whose subquotient, a Jordan pair, is covered by a finite grid, and it produces a grading of the Lie algebra L by the weight lattice of the root system associated to the covering grid. As a corollary one obtains a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$ such that $B = L_n$. In particular, our assumption on B holds for abelian inner ideals of finite length in nondegenerate Lie algebras.

1 Introduction

A *finite $\mathbb{Z}$-grading* of a Lie algebra L over a unital commutative ring Φ is a nontrivial $\mathbb{Z}$-grading with finite support, i.e., there exists a positive natural number n and a family $(L_i)_{-n\le i\le n}$ of Φ-submodules of L such that

$$L = \bigoplus_{i=-n}^{n} L_i, \quad L_{-n} + L_n \neq 0, \quad [L_i, L_j] \subset L_{i+j}$$

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for all i, j with the understanding that $L_{i+j} = 0$ if $|i + j| > n$. In this case, one says that L is $(2n + 1)$-*graded*. Simple Lie algebras that have a $(2n + 1)$-grading and that are defined over a field of characteristic $\geq 4n + 1$ or 0 were classified by Zelmanov [31] up to the description of finite $\mathbb{Z}$-gradings of simple associative algebras with involutions. This description was later given by Smirnov [29, 30].

The main result of this article is a method to construct finite $\mathbb{Z}$-gradings of Lie algebras. Roughly speaking, we show that a sufficiently nice "top" L_n creates a $(2n + 1)$-grading of L.

What are nice "tops"? The submodule L_n of any $(2n+1)$-grading of L is an abelian inner ideal in the sense of Benkart [3], i.e., a Φ-submodule B satisfying $[B, [L, B]] \subset B$ and $[B, B] = 0$. The pair (L_n, L_{-n}) of the "wings" of the $(2n + 1)$-grading is a Jordan pair with respect to the Jordan triple products $\{x, y, z\} = [[x, y], z]$. It is enough to specify the Jordan triple product since we will assume throughout the paper that 2 and 3 are invertible in Φ, and from Section 4 on that 5 too is invertible. It is these two algebraic structures, abelian inner ideals in Lie algebras and Jordan pairs, that form the basis of our approach.

We do not require that we are given submodules L_n, L_{-n} of L. Rather, we associate a Jordan pair S to any abelian inner ideal B of L, which for the case of a nondegenerate $(2n + 1)$-graded L and $B = L_n$ is isomorphic to (L_n, L_{-n}). (We recall that a Lie algebra is *nondegenerate* if $[x, [L, x]] = 0$ implies $x = 0$.) This works as follows. Mimicking the definition of the kernel of an inner ideal in a Jordan pair [18], we define the *kernel* of an abelian inner ideal B in a Lie algebra L as $\operatorname{Ker}_L B = \{x \in L : [B, [x, B]] = 0\}$. Then $S = (B, L/\operatorname{Ker}_L B)$ is a Jordan pair, called the *subquotient* of B, with respect to the Jordan triple products induced by the double commutator of L. That a sufficiently nice "top" L_n creates a $(2n + 1)$-grading of L can now be expressed more precisely.

Theorem 1.1. Let L be a Lie algebra and suppose B is an abelian inner ideal of L whose subquotient S is covered by a finite grid. Then there exists a finite $\mathbb{Z}$-grading, say a $(2n + 1)$-grading, such that $B = L_n$, $\operatorname{Ker}_L B = L_{-n+1} \oplus \cdots \oplus L_n$ and $(L_n, L_{-n}) \cong S$. $\square$

For the nonexpert in Jordan theory we mention that a grid in a Jordan pair is a special family of idempotents, see [22, 26] for details. The assumption on S is for example fulfilled in case the subquotient is a nondegenerate and Artinian Jordan pair, since these can be characterized as those Jordan pairs that are covered by a finite division grid [[18]; Theorem 5.2]. And as we show in Proposition 3.5, the subquotient of an abelian inner ideal is always nondegenerate and Artinian if L itself is nondegenerate and B has finite length, i.e. every proper chain of inner ideals of L contained in B is finite. Let us now discuss some of the techniques and concepts used in the proof of the result.

- **Idempotents:** Idempotents in Jordan pairs are of course a well-known concept. Motivated by the Jordan pair case, we call a pair of elements (e^+, e^-) in $L \times L$, L a Lie algebra, an *idempotent of* L if $(e^+, h_e = [e^+, e^-], e^-)$ is an $\mathfrak{sl}_2$-triple in L and $(\operatorname{ad} e^+)^3 = 0$. Then $(\operatorname{ad} e^-)^3 = 0$ and $\operatorname{ad} h_e$ is diagonalizable with eigenvalues $0, \pm 1, \pm 2$, i.e., $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ for the eigenspaces L_i of $\operatorname{ad} h_e$ (it is here that we need our assumption that 5 is invertible in Φ). As in Jordan theory, the Peirce decomposition of one idempotent can be refined by considering a finite family $\mathcal{E}$ of idempotents in L which is compatible in the sense that $[h_e, h_f] = 0$ for $e, f \in \mathcal{E}$.

  These definitions are well behaved with respect to subquotients: If $\mathcal{E}$ is a compatible family of idempotents in L and B is an abelian inner ideal of L such that $e^+ \in B$ for all $e \in \mathcal{E}$, then the canonical image of $\mathcal{E}$ in the subquotient is a compatible family of idempotents in the Jordan pair sense. It is crucial for our work that we can also go backwards. Indeed, the essence of Proposition 5.4 is that any finite family of compatible idempotents in S can be lifted to a compatible family of idempotents in L. We note that the lifting of a single idempotent is essentially a graded version of the Jacobson-Morozov Lemma.
- **3-graded root systems:** The combinatorics of grids in Jordan pairs is best described using 3-graded root systems, see [[19]; §18]. To any grid $\mathcal{G}$ in a Jordan pair one can associate a 3-graded root system $R = R_1 \cup R_0 \cup R_{-1}$ and an enumeration of the grid as $\mathcal{G} = (g_\alpha : \alpha \in R_1)$ such that the relations between the idempotents in $\mathcal{G}$ are described by the combinatorics (angles) of R. For example, the idempotents g_α, g_β are orthogonal if and only if the roots α, β are orthogonal. In general, the root system R is locally finite, but for Theorems 1.1 and 1.2 we will only be using finite grids and hence finite root systems. For the root system R we denote by $\mathcal{P}(R)$ the abelian group of the weights of R. We recall that $R \subset \mathcal{P}(R)$ canonically. Theorem 1.1 is a corollary of the following result.

Theorem 1.2. Let B be an abelian inner ideal of a Lie algebra L whose subquotient $S = (B, L/\operatorname{Ker}_L B)$ is covered by a finite standard grid $\mathcal{G}$ with associated 3-graded root system $R = R_1 \cup R_0 \cup R_{-1}$.

Then $\mathcal{G}$ lifts to a compatible family $\mathcal{E} = (e_\alpha : \alpha \in R_1)$, $e_\alpha = (e_\alpha^+, e_\alpha^-)$, of idempotents in L whose joint Peirce spaces induce a $\mathcal{P}(R)$-grading of L:

$$L = \bigoplus_{\omega \in \mathcal{P}(R)} L_\omega, \quad \text{where} \quad L_\omega = \{x \in L : [h_\alpha, x] = \langle \omega, \alpha^\vee \rangle x \text{ for all } \alpha \in R_1\}$$

and $h_\alpha = [e_\alpha^+, e_\alpha^-]$. Moreover,

$$B = \bigoplus_{\omega \in R_1} L_\omega, \quad \operatorname{Ker}_L B = \bigoplus_{\omega \notin R_{-1}} L_\omega.$$

The subalgebra $\mathfrak{g}$ generated by all $e_\alpha^\pm$ is R-graded in the sense of [26]. If Φ is a field of characteristic 0 then $\mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra of type R with splitting Cartan subalgebra $\mathfrak{h} = \sum_{\alpha \in R_1} \Phi h_\alpha$ and is isomorphic to the Tits-Kantor-Koecher algebra of the Jordan pair generated by $\mathcal{G}$. $\square$

Our assumption that $\mathcal{G}$ be a standard grid is not serious (but necessary for the second part of Theorem 1.2), since any covering grid can be replaced by a covering standard grid with the same Peirce spaces and associated 3-graded root system. We point out that the $\mathcal{P}(R)$-grading of L constructed above has many of the features of a grading of L by a root systems, as defined by [2], [5] and [26], see 5.2.

The support $\operatorname{supp} L = \{\omega \in \mathcal{P}(R) : L_\omega \neq 0\}$ of the $\mathcal{P}(R)$-grading of L contains R but possible more weights. We construct a group homomorphism $\varphi\colon\ \mathcal{P}(R) \to \mathbb{Z}$ such that for a suitable positive integer n we have $|\varphi(\omega)| \leq n$ for $\omega \in \operatorname{supp} L$ with $\varphi(\omega) = n \Leftrightarrow \omega \in R_1$. One then obtains a $(2n+1)$-grading of L and hence a proof of Theorem 1.1 by putting $L_i = \bigoplus_{\varphi(\omega)=i} L_\omega$ for $-n \leq i \leq n$. We note that in case of an irreducible R, equivalently a simple subquotient S, the number n above can be chosen as the Coxeter number h of R. Namely, in this case we can take $\varphi(\omega) = \sum_{\alpha \in R_1} \langle \omega, \alpha^\vee \rangle$, and we have $\sum_{\alpha \in R_1} \langle \beta, \alpha^\vee \rangle = h$ for all $\beta \in R_1$.

Applications: It is an immediate corollary of Theorem 1.1 that $C = L_{-n}$ is another abelian inner ideal with $\operatorname{Ker}_L C = L_{-n} \oplus \cdots \oplus L_{n-1}$. Thus, the abelian inner ideal B is *complemented* by C in the sense that $L = B \oplus \operatorname{Ker}_L C = C \oplus \operatorname{Ker}_L B$ (Theorem 6.1). This is essential for characterizing Lie algebras in which every inner ideal is complemented [11].

Using the Tits-Kantor-Koecher construction we can give another application of our results, namely to inner ideals in Jordan pairs (Corollary 6.4): If the subquotient of an inner ideal B of a Jordan pair V is covered by a finite grid, it can be lifted to a finite grid in V which induces a finite $\mathbb{Z}$-grading of V. Moreover, B is complemented in the sense of [18].

The paper is organized as follows. After a review of some concepts from the theory of Lie algebras and Jordan pairs in Section 2, we study the kernel and subquotient of an inner ideal in a Lie algebra in Section 3. In Section 4 we review and prove some results for 3-graded root systems. The main work is done in Section 5, in particular in Proposition 5.4 and Theorem 5.5, which together provide a proof of Theorem 1.2. For the applications in Jordan pairs, it is necessary to prove parts of these results in the graded setting. The final Section 6 is devoted to the applications mentioned above. We also discuss there some examples illustrating the relationship between abelian inner ideals and finite $\mathbb{Z}$-gradings of Lie algebras.

2 Preliminaries

2.1 Basic notions

Throughout this article we will be dealing with Lie algebras, Jordan algebras and Jordan pairs over a ring of scalars Φ containing $\mu \cdot 1_\Phi \in \Phi^\times$ for $\mu = 2, 3$ where $\Phi^\times$ denotes the invertible elements of Φ. So both the Jordan algebras and Jordan pairs considered here are *linear*. From Section 5 on we will also assume that $5 \cdot 1_\Phi$ is invertible in Φ.

We will use standard notation. For example, the product in a Lie algebra will be denoted $[x, y]$, while $\operatorname{ad} x$ or ad_x is the adjoint map determined by x. We will also use the abbreviation $[x_1, x_2, \dots, x_{n-1}, x_n] = (\operatorname{ad} x_1)(\operatorname{ad} x_2) \cdots \operatorname{ad} x_{n-1}(x_n)$.

For Jordan pairs $V = (V^+, V^-)$ we will follow the terminology of [16]. In particular, it follows from [[16]; p. 55] that a pair $V = (V^+, V^-)$ of Φ-modules with trilinear maps $\{\cdot\ \cdot\ \cdot\} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $\sigma = \pm$, is a Jordan pair if and only if the triple products satisfy the two conditions:

$$\{x, y, z\} = \{z, y, x\} \tag{J1}$$

$$\{u, v, \{x, y, z\}\} + \{x, \{v, u, y\}, z\} = \{\{u, v, x\}, y, z\} + \{x, y, \{u, v, z\}\}. \tag{J2}$$

2.2 Nondegeneracy and primeness

Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$, and V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for any ideals $B = (B^+, B^-)$, $C = (C^+, C^-)$ of V. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* if $\operatorname{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for any ideals I, J of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate.

2.3 Inner Ideals and Jordan Elements

Given a Jordan pair $V = (V^+, V^-)$, an *inner ideal* of V is any Φ-submodule B of V^σ such that $\{B, V^{-\sigma}, B\} \subset B$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[B, L, B] \subset B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B, B] = 0$. In the following we will mainly consider abelian inner ideals. This is not such a great restriction as it may look at first sight since in a nondegenerate simple Artinian Lie algebra every inner ideal $B \neq L$ is abelian ([[4]; Lemma 1.13]).

For $b \in L$ the following conditions are equivalent [[4]; Lemma 1.8]:

(i) $\mathrm{ad}_b^3 = 0$,

(ii) there exists an abelian inner ideal B containing $b \in B$.

Any element $b \in L$ satisfying these two conditions is called a *Jordan element*. Any Jordan element b gives rise to the abelian inner ideals $[b] := [b, b, L]$ and $(b) := \Phi b + [b]$.

Lemma 2.1. Let I be an ideal of a Lie algebra and $x \in I$ a Jordan element of I. For any $a, b \in I$, we have

(i) $X^2AX = XAX^2$,

(ii) $X^2AX^2 = 0$,

(iii) $\mathrm{ad}^2_{X^2(a)} = X^2A^2X^2$,

(iv) $X^2ABX^2 = X^2BAX^2 = \mathrm{ad}_{X^2(a)}\mathrm{ad}_{X^2(b)}$

where capital letters denote the adjoint maps with respect to those elements. □

Proof. (i), (ii), (iii) follow as in [[4]; Lemma 1.7 (i), (ii), (iii)] since $X^3(a) = 0$. For (iv) we use (ii) and get $X^2ABX^2 = X^2[A, B]X^2 + X^2BAX^2 = X^2\mathrm{ad}_{[a,b]}X^2 + X^2BAX^2 = X^2BAX^2$, and from (iii) that $\mathrm{ad}^2_{X^2(a+b)} = X^2\mathrm{ad}^2_{a+b}X^2$. But $\mathrm{ad}^2_{X^2(a+b)} = X^2A^2X^2 + X^2B^2X^2 + 2\mathrm{ad}_{X^2(a)}\mathrm{ad}_{X^2(b)}$ (since the operators $\mathrm{ad}_{X^2(a)}$ and $\mathrm{ad}_{X^2(b)}$ commute because the inner ideal $[x, [x, I]]$ is abelian), and $X^2\mathrm{ad}^2_{a+b}X^2 = X^2A^2X^2 + X^2B^2X^2 + 2X^2ABX^2$. Hence, $X^2ABX^2 = \mathrm{ad}_{X^2(a)}\mathrm{ad}_{X^2(b)}$, as required. ■

Remark 2.2. Let (V^+, V^-) be a pair of Φ-submodules of a Lie algebra L such that $\{x, y, z\} := [[x, y], z] \in V^\sigma$ for all $x, z \in V^\sigma$ and $y \in V^{-\sigma}$. It is a straightforward consequence of the Jacobi identity that the pair $V = (V^+, V^-)$ satisfies the identity (J2) defining a Jordan pair, but not necessarily the identity (J1). However, both identities are fulfilled for a pair (B, C) of abelian inner ideals, and hence (B, C) is a Jordan pair.

2.4 Gradings

Let L be a Lie algebra and let Γ be an abelian group, written additively. We say that L is *graded by* Γ and call this a Γ-*grading of* L if there exists a decomposition $L = \bigoplus_{\gamma\in\Gamma} L^\gamma$, where the L^γ are Φ-submodules of L, satisfying $[L^\gamma, L^\delta] \subset L^{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. A *finite* $\mathbb{Z}$-*grading* is a nontrivial $\mathbb{Z}$-grading such that the *support set* $\operatorname{supp} L = \{\gamma \in \mathbb{Z} : L_\gamma \neq 0\}$ is finite. Hence $L = L_{-n} \oplus \cdots \oplus L_n$ for some positive integer n. If $L_n + L_{-n} \neq 0$, we will call such a grading a $(2n+1)$-*grading*. Note that if L is nondegenerate then both L_n and L_{-n} are nonzero.

Let $L = \bigoplus_{\gamma\in\Gamma} L^\gamma$ be a Γ-graded Lie algebra. A Φ-submodule M of L will be called a *graded submodule* if $M = \bigoplus_{\gamma\in\Gamma}(M \cap L^\gamma)$, in which case we will write $M = \bigoplus M^\gamma$ where $M^\gamma = M \cap L^\gamma$. An inner ideal B is *graded* if its underlying submodule is graded. We will refer to the elements of L^γ, $\gamma \in \Gamma$, as *homogeneous elements*. We will say that L is *graded-nondegenerate with respect to* Γ if it does not have homogeneous absolute zero divisors.

If Δ is another abelian group, we will say that a Δ-grading $L = \bigoplus_{\delta\in\Delta} L_\delta$ is *compatible* with the given Γ-grading if, putting $L_\delta^\gamma = L^\gamma \cap L_\delta$, we have $L^\gamma = \bigoplus_{\delta\in\Delta} L_\delta^\gamma$ for all $\gamma \in \Gamma$, or equivalently $L_\delta = \bigoplus_{\gamma\in\Gamma} L_\delta^\gamma$ for all $\delta \in \Delta$. Of course, two compatible Γ- and Δ-gradings are the same as a $\Gamma \oplus \Delta$-grading, but it is usually more instructive to keep the two gradings apart.

Similarly, a Γ-*grading of a Jordan pair* $V = (V^+, V^-)$ consists of decompositions $V^\sigma = \bigoplus_{\gamma\in\Gamma} V_\gamma^\sigma$ of V^σ with V_γ^σ being Φ-submodules of V^σ such that $\{V_\gamma^\sigma, V_\delta^{-\sigma}, V_\epsilon^\sigma\} \subset V_{\gamma+\delta+\epsilon}^\sigma$ for all $\gamma, \delta, \epsilon \in \Gamma$. Graded inner ideals of a Γ-graded Jordan pair and homogeneous elements are defined analogously to the case of Lie algebras. The proof of the following lemma is a simple verification, left to the reader.

Lemma 2.3. Let $L = \bigoplus_{\gamma\in\Gamma} L^\gamma$ be a Γ-graded Lie algebra with a compatible $(2n+1)$-grading $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$. Then $V = (L_n, L_{-n})$ is a Γ-graded Jordan pair with respect to $(V^\pm)^\gamma = L_{\pm n}^\gamma$ and the triple products $\{x, y, z\}$ defined in Remark 2.2. Moreover,

(i) a Φ-submodule B of $L_{\pm n}$ is an abelian inner ideal of L if and only if it is an inner ideal of V, and

(ii) if L is graded-nondegenerate with respect to Γ or nondegenerate, then so is the Jordan pair V. □

2.5 Socle and chain conditions

(i) Recall that the *socle* of a nondegenerate Jordan pair V is $\operatorname{Soc} V = (\operatorname{Soc} V^+, \operatorname{Soc} V^-)$ where $\operatorname{Soc} V^\sigma$ is the sum of all minimal inner ideals of V contained in V^σ [17]. The *socle* of a nondegenerate Lie algebra L is Soc$L$, defined as the sum of all minimal inner ideals of L [7].

(ii) By [[17]; Theorem 2] for the Jordan pair case and [[7]; Theorem 3.6] for the Lie case, the socle of a nondegenerate Jordan pair or Lie algebra is the direct sum of its simple ideals. Moreover, each simple component of Soc$L$ is either inner simple or contains an abelian minimal inner ideal [[4]; Theorem 1.12].

(iii) A Lie algebra L or Jordan pair V is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals. While any nondegenerate Artinian Jordan pair coincides with its socle (by the elemental characterization of the socle, [[17]; Theorem 1]), for a nondegenerate Artinian Lie algebra L we only have that L has an essential socle in the sense that every nonzero ideal has a nonzero intersection with the socle [[7]; Corollary 3.7 and Remark 3.8].

3 Kernels and subquotients

3.1 Kernels

Let $V = (V^+, V^-)$ be a linear Jordan pair and $B \subset V^+$ an inner ideal of V. Following [18], the *kernel* of B is the set $\mathrm{Ker}_V B = \{x \in V^- : \{B, x, B\} = 0\}$. Then $(0, \mathrm{Ker}_V B)$ is an ideal of the Jordan pair (B, V^-) and the quotient $S = (B, V^-)/(0, \mathrm{Ker}_V B) = (B, V^-/\mathrm{Ker}_V B)$ is called the *subquotient of V with respect to B*. The kernel and the corresponding subquotient of an inner ideal $B \subset V^-$ are defined similarly.

The analogous versions of all of these results hold for inner ideals in Lie algebras, if we replace the Jordan triple product $\{x, y, z\}$ by the left double commutator $[[x, y], z]$, cf. Remark 2.2.

Definition 3.1. Let B be an inner ideal of a Lie algebra L. The *kernel* of B is the Φ-submodule $\mathrm{Ker}_L B = \{x \in L : [B, B, x] = 0\}$.

In the following lemma we will consider the pair (L, L) of a Lie algebra L with respect to the triple products of Remark 2.2. We will use the concepts of subpairs, ideals and quotients which are defined in an obvious way, see [[16]; 1.3] for the case of Jordan pairs.

Lemma 3.2. Let B be a Φ-submodule of the Lie algebra L. Then (B, L) is a subpair of (L, L) if and only if B is an inner ideal of L.

In the remainder of this proposition we will assume that B is an inner ideal of L and consider the subpair (B, L) of (L, L). Then the following holds.

(a) $(0, \mathrm{Ker}_L B)$ is the largest among all ideals $I = (I^+, I^-)$ of the pair (B, L) such that $I^+ = 0$. Moreover, $K = \mathrm{Ker}_L B$ satisfies

$$[K, L, B] + [L, B, K] + [B, K, L] \subset K. \tag{3.1}$$

Hence the pair $S = (B, L/\mathrm{Ker}_L B)$, called the *subquotient of* B, has well-defined triple products

$$\{m\,\bar{x}\,n\} = [[m, x], n] \quad \text{and} \quad \{\bar{x}\,m\,\bar{y}\} = \overline{[[x, m], y]}$$

where $m, n \in B$, $x, y \in L$ and $L \to L/\mathrm{Ker}_L B : x \to \bar{x}$ is the canonical map.

(b) S always satisfies the 5-linear identity (J2), and is a Jordan pair if B is an abelian inner ideal.

(c) If B is an abelian inner ideal then $[B, L] \subset \mathrm{Ker}_L B$ and $\mathrm{Ker}_L B = \{x \in L : [b, b, x] = 0 \text{ for all } b \in B\}$.

(d) Assume L is a Γ-graded Lie algebra and B is a graded abelian inner ideal. Then $\mathrm{Ker}_L B$ is a Γ-graded Φ-submodule, and S is a Γ-graded Jordan pair with respect to the quotient grading induced by the Γ-grading of L. □

The proof is a straightforward exercise which will be left to the reader.

Proposition 3.3. Let $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ be a $(2n+1)$-grading of a Lie algebra L with associated Jordan pair $V = (L_n, L_{-n})$.

(i) Let $B \subset L_n$ be an inner ideal of V. Then the kernel of the abelian inner ideal B of L is

$$\mathrm{Ker}_L B = \mathrm{Ker}_V B \oplus L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n.$$

(ii) If L is nondegenerate, then

$$\mathrm{Ker}_L L_n = L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n,$$

and the Jordan pairs (L_n, L_{-n}) and $(L_n, L/\mathrm{Ker}_L L_n)$ are isomorphic.

In particular, any nondegenerate Jordan pair $V = (V^+, V^-)$ is a subquotient, namely isomorphic to the subquotient of its Tits-Kantor-Koecher algebra with respect to the inner ideal V^+ of V. □

Proof. (i) As pointed out in Lemma 2.3 (i), B is an abelian inner ideal of L, and it is easy to see that $\mathrm{Ker}_L B = \mathrm{Ker}_V B \oplus L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$. If L is nondegenerate, then so is the Jordan pair $V = (L_n, L_{-n})$ and hence $\mathrm{Ker}_V L_n = 0$ by [[18]; 1.4]. Now (ii) follows easily from (i). ■

Remark 3.4. By definition, a properly ascending chain $0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$ of inner ideals of a Lie algebra L has length n. The *length* of an inner ideal B is the supremum of the lengths of chains of inner ideals of L contained in B.

The following elemental characterization of strong primeness for Lie algebras [[12]; Theorem 1.6] will be used in the proof of our next result: A Lie algebra L (over an arbitrary ring of scalars) is strongly prime (as defined in 2.2) if and only if $[x, [y, L]] = 0$ implies $x = 0$ or $y = 0$, for every $x, y \in L$.

Proposition 3.5. Let B be an abelian inner ideal of a Lie algebra L, $K = \mathrm{Ker}_L B$ the kernel of B, and $V = (B, L/K)$ the subquotient of L relative to B.

(i) A Φ-submodule of B is an inner ideal of L if and only if it is an inner ideal of V.

(ii) If C is an inner ideal of L, then $\overline{C} = (C + K)/K$ is an inner ideal of V.

(iii) If L is nondegenerate (strongly prime), then V is nondegenerate (strongly prime).

If L is nondegenerate, then,

(iv) V has nonzero socle if and only if B contains minimal inner ideals. In fact, $\mathrm{Soc}B = \mathrm{Soc}L \cap B$, and

(v) B has finite length if and only if V is Artinian. In this case, $B \subset \mathrm{Soc}L$ and $V \cong (B, I/\mathrm{Ker}_I B)$, where I is any ideal of L containing B.

(vi) If L is strongly prime and B is nonzero and of finite length, then V is a simple nondegenerate Artinian Jordan pair. □

Proof.

(i) is trivial.

(ii) is trivial.

(iii) Suppose that L is strongly prime. If $\{b, L/K, b'\} = 0$ for some $b, b' \in B$, then $[[b, L], b'] = 0$ and hence $b = 0$ or $b' = 0$ by the elemental characterization of strong prime Lie algebras in Remark 3.4. Suppose now that $\{\overline{a}, B, \overline{c}\} = 0$ for some $a, c \in L$. i.e., $[[a, B], c] \subset K$. By Lemma 2.1 (iv), we have for any $b \in B$

$$0 = [b, b, a, c, B] \supset [b, b, a, c, b, b, L] = [[b, b, a], [b, b, c], L],$$

which, again by the elemental characterization of strong primeness in Remark 3.4 implies $[b, [b, a]] = 0$ or $[b, [b, c]] = 0$, i.e., $\bar{a} = 0$ or $\bar{c} = 0$. A similar argument applies when L is nondegenerate to yield nondegeneracy of V.

(iv) By (iii) V is nondegenerate, and by (i) the minimal inner ideals of V which are contained in B are those minimal inner ideals of L which are contained in B.

(v) By [[18]; Corollary 4.8], B has finite length if and only if V is Artinian. In this case, $B = \operatorname{Soc} B \subset \operatorname{Soc} L$, since Artinian nondegenerate Jordan pairs coincide with their socles. Let I be an ideal of L containing B. The injection $j\colon I \to L$ induces the Jordan pair monomorphism $(1_B, \bar{j})\colon (B, I/\mathrm{Ker}_I B) \to (B, L/\mathrm{Ker}_L B)$, but since Artinian nondegenerate Jordan pairs are von Neumann regular, $(1_B, \bar{j})$ is actually an isomorphism: $L/K = \{L/K, B, L/K\} = \overline{[[L, B], L]} = \bar{I}$.

(vi) By (iii) and (v), V is a strongly prime Artinian Jordan pair. Hence, by the socle structure theorem, see 2.5, $V = \operatorname{Soc} V$ is a simple Jordan pair. ■

3.2 Jordan algebras of a Lie algebra

In the recent paper [10], the first three authors of this article showed how to attach a Jordan algebra L_x to any Jordan element x of a Lie algebra L (over a ring of scalars Φ containing $\frac{1}{6}$). We will show that L_x can be regarded as the x-homotope of the subquotient of L relative to the abelian inner ideal $B = (x) = \Phi x + [x, x, L]$. To do so, the following facts will be used.

Remark 3.6.

(i) Let $V = (V^+, V^-)$ be a Jordan pair and $x \in V^{-\sigma}$, $\sigma = \pm$. The Φ-module V^σ becomes a Jordan algebra with respect to the product $a \bullet b := \frac{1}{2}\{a, x, b\}$, called the *x-homotope* of V and denoted by $V^{(x)}$ [[16]; 1.9]. Its U-operator is $U_a = Q_a Q_x$.

(ii) Let $B = (x)$ be the (abelian) inner ideal generated by a Jordan element x of L, and put $V = (B, L/\mathrm{Ker}_L B)$. Then $V^{(x)}$ is the Jordan algebra defined on the Φ-module $L/\mathrm{Ker}_L B$ with product $\overline{a} \bullet \overline{b} = \frac{1}{2}\overline{[[a, x], b]}$.

(iii) Actually, the definition of Jordan algebra at a Jordan element given in [10] is slightly different from that of (ii): $\ker_L x := \{z \in L : [x, [x, z]] = 0\}$ is used there instead of $\mathrm{Ker}_L B$. Nevertheless, both definitions agree in the nondegenerate case.

Lemma 3.7. Let L be a nondegenerate Lie algebra and let $x \in L$ be a Jordan element. Then $\mathrm{Ker}_L[x] = \mathrm{Ker}_L(x) = \ker_L x$. □

Proof. Let $z \in \mathrm{Ker}_L\,[x]$. By Lemma 2.1 (iii) we have for every $a \in L$ that

$$0 = [[x, x, a], [[x, x, a], z]] = \mathrm{ad}^2_{\mathrm{ad}^2_x a} z = \mathrm{ad}^2_x \mathrm{ad}^2_a \mathrm{ad}^2_x z$$

which implies $U_{\bar a}\bar z = 0$ for every $\bar a \in L_x$, the Jordan algebra of L at x. But L_x is nondegenerate by [[10]; Proposition 2.15], and hence $U_{L_x}\bar z = 0$ implies $\bar z = 0$, i.e., $\mathrm{ad}^2_x z = 0$, which proves the equality $\mathrm{Ker}_L[x] = \ker_L x$.

Let $z \in \mathrm{Ker}_L\, x$. For any $\lambda \in \Phi$ and $a \in L$, we have $[\lambda x + [x, x, a], [\lambda x + [x, x, a], z]] = \lambda^2[x, x, z] + 2\lambda[[x, x, a], [x, z]] + [[x, x, a], [x, x, a], z]$ where every summand is zero since $\mathrm{ad}^3_x = 0$ and $\mathrm{ad}^2_x z = 0$. Thus, $z \in \mathrm{Ker}_L(x)$. The reverse inclusion $\mathrm{Ker}_L\,(x) \subset \mathrm{Ker}_L[x]$ is trivial. ■

4 Some results on 3-graded root systems

In this section we will state and prove some results on 3-graded root systems. The first result holds for locally finite root systems as studied in [19] and deals with the coroot system $R^\vee$ of R ([[19]; 4.9]), the root lattice $\mathcal{Q}(R)$ and the weight lattice $\mathcal{P}(R)$ of R ([[19]; §7]). We will use special elementary configurations (triangles, quadrangles and diamonds) defined in [[19]; §18]. Following the convention of [19] we will always assume $0 \in R$.

Proposition 4.1. Let (R, R_1) be a 3-graded root system with coroot system $R^\vee$.

(a) The root lattice $\mathcal{Q}(R^\vee)$ of $R^\vee$ is isomorphic to the abelian group presented by generators $\check{x}_\alpha$, $\alpha \in R_1$, and relations

(i) $\check{x}_\alpha = \check{x}_\beta + \check{x}_\gamma$ for all triangles $(\alpha; \beta, \gamma) \subset R_1$, and

(ii) $\check{x}_\alpha + \check{x}_\gamma = \check{x}_\beta + \check{x}_\delta$ for all quadrangles $(\alpha, \beta, \gamma, \delta) \subset R_1$.

(b) A function $\omega\colon\ R_1 \to \mathbb{Z}$ extends to a weight $\tilde{\omega}$ of R if and only if ω satisfies

(i) $\omega(\alpha) = \omega(\beta) + \omega(\gamma)$ for all triangles $(\alpha; \beta, \gamma) \subset R_1$, and

(ii) $\omega(\alpha) + \omega(\gamma) = \omega(\beta) + \omega(\delta)$ for all quadrangles $(\alpha, \beta, \gamma, \delta) \subset R_1$.

In this case, the extension $\tilde{\omega}$ is unique and ω also satisfies

(iii) $2\omega(\alpha) + \omega(\gamma) = \omega(\beta) + \omega(\delta)$ for all diamonds $(\alpha; \beta, \gamma, \delta) \subset R_1$. □

The proof is essentially an application of [[19]; Proposition 11.12] with P the parabolic subset $(R_0 \cup R_1)^\vee$ of $R^\vee$, while (b) follows immediately from (a). Details will be contained in [20].

Definition 4.2. Let (R, R_1) be a finite 3-graded root system. Recall that any root α gives rise to a unique weight $\tilde{\alpha}$ defined by $\tilde{\alpha}(\beta) = \langle \alpha, \beta^\vee \rangle$ for $\beta \in R$. We will identify $\tilde{\alpha} = \alpha$. For $\omega \in \mathcal{P}(R)$ and for an orthogonal system $\mathcal{O} \subset R_1$ we define

$$\tau(\omega) = \textstyle\sum_{\alpha \in R_1} \langle \omega, \alpha^\vee \rangle \quad \text{and} \quad \tau_{\mathcal{O}}(\omega) = \textstyle\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle.$$

Proposition 4.3. Let (R, R_1) be a finite 3-graded root system. Then there exist a positive integer n and a group homomorphism $\varphi\colon\ \mathcal{P}(R) \to \mathbb{Z}$ such that

(i) $\varphi(\alpha) = n$ for $\alpha \in R_1$, and

(ii) $|\varphi(\omega)| < n$ for every $\omega \in \mathcal{P}(R)$ satisfying $|\tau_{\mathcal{O}}(\omega)| \leq 1$ for every orthogonal system $\mathcal{O} \subset R_1$. □

If R is irreducible then τ satisfies (i) and (ii) with n the Coxeter number of R.

We will first consider an irreducible R and show that $\varphi = \tau$ satisfies (i) and (ii). The general case will then be dealt with in 4.8. In the irreducible case we will use the classification of irreducible 3-graded root systems, as given in [[19]; 17.8 and 17.9]. This will also give us some more precise information about $\tau(\omega)$. We let h denote the Coxeter number of R, which can be found in the tables of [[6]; VI].

4.1 Rectangular grading A_l^p, $1 \leq p \leq [\frac{l+1}{2}]$

Here $R = \mathrm{A}_l$ so the Coxeter number $h = l + 1$. Let $q = l + 1 - p$. Up to isomorphism we can assume that the 1-part R_1 of this 3-grading is given by $R_1 = \{\epsilon_i - \epsilon_j : 1 \leq i \leq p < j \leq h\}$.

It is then easily seen that $\tau(\beta) = h$ for every $\beta \in R_1$. Moreover, R_1 is a disjoint union of q orthogonal systems $\mathcal{O}_i$ of length p, whence $|\tau(\omega)| \le \sum_{i=1}^{q} |\tau_{\mathcal{O}_i}(\omega)| \le q < h$ for ω as in Proposition 4.3 (ii).

4.2 Odd quadratic form grading $\mathrm{B}_l^{\mathrm{qf}}$

Here $R = \mathrm{B}_l$, $l \ge 2$, so $h = 2l$. Up to isomorphism, the 1-part of R_1 of this 3-grading is $R_1 = \{\epsilon_1\} \cup \{\epsilon_1 \pm \epsilon_i : 2 \le i \le l\}$. Each $(\epsilon_1; \epsilon_1 + \epsilon_i, \epsilon_1 - \epsilon_i)$ is a triangle. It then follows from Proposition 4.1 (b.i) that $\tau(\omega) = l\langle \omega, \epsilon_1^\vee \rangle$ for any $\omega \in \mathcal{P}(R)$. In particular, $\tau(\alpha) = 2l$ for $\omega = \alpha \in R_1$ and $\tau(\omega) \in \{0, \pm l\}$ for any $\omega \in \mathcal{P}(R)$ satisfying Proposition 4.3 (ii).

4.3 Hermitian grading $\mathrm{C}_l^{\mathrm{her}}$

Here $R = \mathrm{C}_l$, $l \ge 3$, so $h = 2l$. Up to isomorphism, the 1-part R_1 of this 3-grading is given by $R_1 = \{\epsilon_i + \epsilon_j : 1 \le i,j \le l\}$. For $i \ne j$, the family $(\epsilon_i + \epsilon_j : 2\epsilon_i, 2\epsilon_j)$ is a triangle, whence $(\epsilon_i + \epsilon_j)^\vee = (2\epsilon_i)^\vee + (2\epsilon_j)^\vee$. It then follows that

$$\tau(\omega) = l\left(\sum_{i=1}^{l} \langle \omega, (2\epsilon_i)^\vee \rangle\right)$$

holds for any $\omega \in \mathcal{P}(R)$. In particular $\tau(\beta) = 2l$ for any $\beta \in R_1$, while $\tau(\omega) \in \{0, \pm l\}$ for ω as in Proposition 4.3 (ii).

4.4 Even quadratic form grading $\mathrm{D}_l^{\mathrm{qf}}$

Here $R = \mathrm{D}_l$, $l \ge 4$, and $h = 2(l-1)$. Up to isomorphism $R_1 = \{\epsilon_1 \pm \epsilon_i : 1 < i \le l\}$. Since $(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_i, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_i)$ is a quadrangle for $2 < i \le l$, we get $\tau(\omega) = (l-1)\big(\langle \omega, (\epsilon_1 + \epsilon_2)^\vee \rangle + \langle \omega, (\epsilon_1 - \epsilon_2)^\vee \rangle\big)$ for every $\omega \in \mathcal{P}(R)$. This easily implies (i) and (ii) of Proposition 4.3.

4.5 Alternating grading $\mathrm{D}_l^{\mathrm{alt}}$

Here $R = \mathrm{D}_l$, $l \ge 4$ and $h = 2(l-1)$. We abbreviate $(ij) = \epsilon_i + \epsilon_j$. Up to isomorphism we then have $R_1 = \{(ij) : 1 \le i < j \le l\}$. We first consider the case of an even l. Then $\mathcal{O} = \{(12), (34), \dots (l-1, l)\}$ is an orthogonal system such that $R_1 = \mathcal{O} \cup \bigcup_{i=1}^{l-2} \mathcal{O}_i$ where each $\mathcal{O}_i$ is an orthogonal system of two roots with the property that each $\mathcal{O}_i$ together with two roots of $\mathcal{O}$ forms a quadrangle. This implies $\tau(\omega) = (l-1)\sum_{\alpha\in\mathcal{O}}\langle\omega,\alpha^\vee\rangle$. If l is odd, we apply the previous considerations to the 3-graded subsystem with 1-part $\{(ij) : 1 \le i < j \le l-1\}$, and get that $\tau(\omega) = (l-2)\big(\sum_{\alpha\in\mathcal{O}}\langle\omega,\alpha^\vee\rangle\big) + \sum_{i=1}^{l-1}\langle\omega,(il)^\vee\rangle$. In both cases, (i) and (ii) of Proposition 4.3 easily follow.

4.6 Bi-Cayley grading $\mathrm{E}_6^{\mathrm{bi}}$

Here $R = \mathrm{E}_6$ and $h = 12$. By [23] the 1-part of this 3-grading is cog-isomorphic with the 16 tripotents of a bi-Cayley grid $\mathcal{B}$ in a Jordan triple system as defined in [[22]; III, Section 3.1]. By definition, a cog-isomorphism is a bijection which preserves the elementary relations (orthogonality, collinearity and governing) in R_1 and in $\mathcal{B}$. In particular, it follows from [[22]; III, Section 3.1] that, letting $e_i^\pm \in \mathcal{B}$ correspond to $\alpha_i^\pm \in R_1$, the 1-part $R_1 = (\alpha_i^\sigma : \sigma = \pm, 1 \le i \le 8)$ is the union of the 1-parts of two $\mathrm{D}_5^{\mathrm{qf}}$-gradings, namely $(\alpha_i^\sigma : \sigma = \pm, 1 \le i \le 4)$ and $(\alpha_i^\pm : \sigma = \pm, 5 \le i \le 8)$. By 4.4 we therefore have $\tau(\omega) = 4\big(\langle\omega,(\alpha_1^+)^\vee\rangle + \langle\omega,(\alpha_1^-)^\vee\rangle + \langle\omega,(\alpha_5^+)^\vee\rangle + \langle\omega,(\alpha_5^-)^\vee\rangle\big)$ for any $\omega \in \mathcal{P}(R)$, which implies (ii) of Proposition 4.3. That also (i) holds then follows from the Peirce relations in the bi-Cayley grid $\mathcal{B}$.

4.7 Albert grading $\mathrm{E}_7^{\mathrm{Alb}}$

Here $R = \mathrm{E}_7$ and $h = 18$. We will proceed as in 4.6. The 1-part R_1 of this 3-grading is cog-isomorphic to the 27 tripotents of an Albert grid in a Jordan triple system. The structure of the Albert grid ([[22]; III, Section 3.2]) then shows that R_1 contains an orthogonal system $(\alpha_1,\alpha_2,\alpha_3)$ such that $R_1 \setminus \{\alpha_1,\alpha_2,\alpha_3\} = \bigcup_{i=1}^{12}\mathcal{O}_i$, where each $\mathcal{O}_i = \{\beta_I^+,\beta_i^-\}$ is an orthogonal system such that $(\beta_i^+,\alpha_j,\beta_i^-,\alpha_k)$ is a quadrangle for a unique pair $j,k \in \{1,2,3\}$. The Peirce relations in the Albert grid show that $\tau(\omega) = 9\sum_{i=1}^{3}\langle\omega,\alpha_i^\vee\rangle$ for any $\omega \in \mathcal{P}(R)$ and that (i) and (ii) of Proposition 4.3 hold.

4.8 Proof of Proposition 4.3

We have seen in 4.1–4.7 that Proposition 4.3 holds for an irreducible root system. Let $R = \bigcup_{i=1}^{s} R^{(i)}$ be the decomposition of R into its irreducible components, let $n = \mathrm{lcm}(h_1,\ldots,h_n)$ where h_i is the Coxeter number of the irreducible component $R^{(i)}$ and let τ_i be the function of Definition 4.2 for $R^{(i)}$. We claim that

$$\varphi(\omega) = \sum_{i=1}^{s}\frac{n}{h_i}\,\tau_i(\omega)$$

fulfills (i) and (ii) of Proposition 4.3. Obviously $\varphi\colon \mathcal{P}(R) \to \mathbb{Z}$ is a group homomorphism. For $\beta \in R_1 \cap R^{(i)}$ we have $\varphi(\beta) = \frac{n}{h_i}\tau_i(\beta) = n$. For ω as in (ii) note first that the number of irreducible components on which ω does not vanish is at most two. Indeed, if $\langle \omega, \alpha_i^\vee \rangle \neq 0$ for α_i, $i = 1,2,3$, belonging to different components, then $(\alpha_1, \alpha_2, \alpha_3)$ is an orthogonal system. Because $|\langle \omega, \alpha_i^\vee \rangle| = 1$ there exists $\{i,j\} \subset \{1,2,3\}$ such that $\langle \omega, \alpha_i^\vee \rangle + \langle \omega, \alpha_j^\vee \rangle = \pm 2$, contradiction. The same argument also shows that if ω does not vanish on two irreducible components then ω has nonnegative values on one component, say on $R^{(i)}$, and nonpositive values on the second component, say on $R^{(j)}$. We therefore get

$$\varphi(\omega) = \tfrac{n}{h_i}\,\tau_i(\omega) + \tfrac{n}{h_j}\,\tau_j(\omega) < \tfrac{n}{h_i}\,\tau_i(\omega) < \tfrac{n}{h_i}\,h_i = n,$$

and similarly $\varphi(\omega) > \frac{n}{h_j}\tau_j(\omega) > -n$.

5 Lifting of idempotents

From now on we assume that all modules, and hence all Lie algebras and Jordan pairs are defined over a ring of scalars Φ with $\mu 1_\Phi \in \Phi^\times$ for $\mu = 2,3,5$.

Lemma 5.1. Let $S = \{-2,-1,0,1,2\}$, let M be a Φ-module and suppose $H, F \in \mathrm{End}_\Phi M$ satisfy $\prod_{\sigma \in S}(H - \sigma) = 0$ and $[H,F] = -2F$, where we abbreviated $H - \sigma = H - \sigma \mathrm{Id}_M$ for $\sigma \in S$. Then $F^3 = 0$. □

Proof. If $\sigma, \tau \in S$ with $\sigma \neq \tau$ then $0 < |\sigma - \tau| \leq 4$, whence $(\sigma - \tau)1_\Phi \in \Phi^\times$. Therefore M has an eigenspace decomposition $M = \bigoplus_{\sigma \in S} M_\sigma$ where $M_\sigma = \mathrm{Ker}(H - \sigma)$ [[4]; Lemma 2.1]. From $[H,F] = -2F$ we get $F^l M_\sigma \subset M_{\sigma - 2l}$ for any $l \in \mathbb{N}$. We claim

$$F^l M_{-4+2l} = 0 = F^l M_{-3+2l} \quad \text{for } 1 \leq l \leq 3. \tag{5.1}$$

Indeed, $F^l M_{-4+2l} \subset M_{-4}$ and $F^l M_{-3+2l} \subset M_{-3}$. Since $(S+4)1_\Phi = \{2,3,\dots,6\}1_\Phi \subset \Phi^\times$ and $(S+3)1_\Phi = \{1,\dots,5\}1_\Phi \subset \Phi^\times$, it follows that $M_{-4} = 0 = M_{-3}$, proving (5.1). Because any $\sigma \in S$ is of the form $\sigma = -4 + 2l$ or $\sigma = -3 + 2l$ for suitable $l \in \{1,2,3\}$, we get $F^3 M_\sigma = F^{3-l} F^l M_\sigma = 0$ by (5.1), and $F^3 = 0$ follows. ■

Proposition 5.2. Let Γ be an abelian group, $L = \bigoplus_{\gamma \in \Gamma} L^\gamma$ a Γ-graded Lie algebra. If $0 \neq e \in L^\alpha$, $\alpha \in \Gamma$, satisfies $(\mathrm{ad}\, e)^3 = 0$ and $[[e,u],e] = 2e$ for some $u \in L^{-\alpha}$, then there exists $v \in L^{-\alpha} \cap \mathrm{Ker}\,\mathrm{ad}\, e$ such that $(e, [e,f], f)$, $f = u - v$, is an $\mathfrak{sl}_2$-triple with $(\mathrm{ad} f)^3 = 0$. □

For $\Gamma = \{0\}$ the result is proven in [[28]; Lemma V.8.2] (Φ a field) and in [[7]; Lemma 2.9]. Our proof is an easy adaptation of Seligman's proof, which — as Seligman states — "is really a summary of certain results of Jacobson".

Proof. We put $h = [e, u] \in L^0$ and thus have $[h, e] = 2e$. Let $E = \operatorname{ad} e \in \operatorname{End}_\Phi L$ and $H = \operatorname{ad} h$. Since E is homogeneous, $\operatorname{Ker} E$ is a Γ-graded submodule: $\operatorname{Ker} E = \bigoplus_{\gamma \in \Gamma} (\operatorname{Ker} E \cap L^\gamma)$. One proves as in [[15]; p. 99] that $H(\operatorname{Ker} E) \subset \operatorname{Ker} E$ and that $H(H-1)(H-2)|\operatorname{Ker} E = 0$, whence $H(\operatorname{Ker} E \cap L^\gamma) \subset \operatorname{Ker} E \cap L^\gamma$ and $H(H-1)(H-2)|(\operatorname{Ker} E \cap L^\gamma) = 0$ for all $\gamma \in \Gamma$. For $0 \le i \ne j \le 2$ we have $(i-j)1_\Phi \in \Phi^\times$. Therefore $H|(\operatorname{Ker} E \cap L^\gamma)$ is diagonalizable with eigenvalues $0, 1_\Phi$ and $2 \cdot 1_\Phi$. It then follows that $H + 2|(\operatorname{Ker} E \cap L^\gamma)$ is invertible for all $\gamma \in \Gamma$. Since $[e, [h, u]] = -2[e, u]$ we have $[h, u] + 2u \in \operatorname{Ker} E \cap L^{-\alpha}$. Hence there exists $v \in L^{-\alpha} \cap \operatorname{Ker} E$ such that $[h, u] + 2u = [h, v] + 2v$. It follows that $(e, h, f), f = u - v$, is an $\mathfrak{sl}_2$-triple. Then $\prod_{j=1}^5 (H - 3 + j) = 0$ by [[14]; Lemma 1]. Finally Lemma 5.1 shows $(\operatorname{ad} f)^3 = 0$. ■

5.1 Compatible Families of Idempotents

We say that $(e^+, e^-) \in L \times L$ is an *idempotent* in L if $[[e^\sigma, e^{-\sigma}], e^\sigma] = 2e^\sigma$ for $\sigma = \pm$, and $[e^+, e^+, e^+, L] = 0$. For an idempotent $e = (e^+, e^-)$, we always let $h_e = [e^+, e^-]$. It is known ([[14]; Lemma 1]) that ad_{h_e} is diagonalizable with eigenvalues $0, \pm 1, \pm 2$, so by Lemma 5.1 also $[e^-, e^-, e^-, L] = 0$. Thus (e^+, h_e, e^-) is an $\mathfrak{sl}_2$-triple with $(\operatorname{ad} e^\sigma)^3 = 0$, and

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad \text{where } L_i = L_i(h_e) = \{x \in L : [h_e, x] = ix\}. \tag{5.2}$$

Following the concepts used in the theory of Jordan pairs, we define the *Peirce spaces* of an idempotent $e = (e^+, e^-) \in \mathcal{L} = L \times L$ by

$$\mathcal{L}_i = \mathcal{L}_i(h_e) = (L_i, L_{-i}), \quad \text{for } i \in \{0, \pm 1, \pm 2\}$$

and call (5.2) the *Peirce decomposition of e*. We note that it is a 3- or 5-grading with $e \in \mathcal{L}_2(h_e)$.

A family $\mathcal{E} = (e_\alpha)_{\alpha \in A}$ of idempotents in L is called *compatible* if $[h_e, h_f] = 0$ for all $e, f \in \mathcal{E}$, and *Peirce-compatible* if every $e \in \mathcal{E}$ lies in a Peirce space of every $f \in \mathcal{E}$. A Peirce-compatible family is easily seen to be compatible.

Any family $\mathcal{E} = (e_\alpha)_{\alpha \in A}$ of idempotents gives rise to *joint Peirce spaces*

$$L_\omega = L_\omega(\mathcal{E}) = \bigcap\nolimits_{\alpha \in A} L_{\omega(\alpha)}(h_\alpha) = \{x \in L : [h_\alpha, x] = \omega(\alpha)x \text{ for all } \alpha \in A\},$$

where $\omega = (\omega(\alpha))_{\alpha\in A} \in \{0, \pm1, \pm2\}^A$, and $h_\alpha = [e_\alpha^+, e_\alpha^-]$. For simplicity we will just write $\omega \in \mathbb{Z}^A$. A compatible family $\mathcal{E}$ is called *toral* if

$$L = \bigoplus_{\omega\in\mathbb{Z}^A} L_\omega. \tag{5.3}$$

Since $|\omega(\alpha)| \leq 2$ it follows that this decomposition is a $\mathbb{Z}^A$-grading of L. It is easily seen that every finite compatible family is toral.

If $L = \bigoplus_{\gamma\in\Gamma} L^\gamma$ is a Γ-graded Lie algebra, an idempotent $e = (e^+, e^-)$ will be called *homogeneous* if $e^+ \in L^\gamma$ and $e^- \in L^{-\gamma}$ for some $\gamma \in \Gamma$. Since then $h_e \in L^0$ the Peirce decomposition (5.2) of e is compatible with the given Γ-grading. More generally, the decomposition (5.3) of a toral family $\mathcal{E}$ of homogeneous idempotents of L is a $\mathbb{Z}^A$-grading which is compatible with the given Γ-grading. We will also use the analogous concepts for a Γ-graded Jordan pair V. For example, if $V^\sigma = \bigoplus_{\gamma\in\Gamma} V_\gamma^\sigma$ an idempotent $e = (e^+, e^-)$ of V is *homogeneous* if $e \in V_\gamma^+$ and $e^- \in V_{-\gamma}^-$ for some $\gamma \in \Gamma$.

Lemma 5.3. Let B be an abelian inner ideal of L, and let $f = (f^+, f^-)$ be an idempotent in L with $f^+ \in B$, thus $L = \bigoplus_{i=-2}^{2} L_i(h_f)$. Then

(a) $[f^+, f^+, f^-, f^-, x_2] = 4x_2$ for any $x_2 \in L_2(h_f)$, and

(b) $\mathrm{Ker}_L B \cap L_{-2}(h_f) = 0$. □

Proof.

(a) From the Jacobi identity we get $[f^+, f^+, f^-, f^-, x_2] = [f^+, h_f, f^-, x_2] + [f^+, f^-, f^+, f^-, x_2] = 0 + [f^+, f^-, h_f, x_2] = 2[f^+, f^-, x_2] = 4x_2$, since $L_i(h_f)$ is the i-eigenspace of $\mathrm{ad}\, h_e$.

(b) Let $y \in \mathrm{Ker}_L B \cap L_{-2}(h_f)$. Since $L_{-2}(h_f) = [f^-, f^-, L_2(f)]$, we can write $y = [f^-, f^-, x_2]$ for some $x_2 \in L_2(f)$. Then $0 = [f^+, f^+, y] = [f^+, f^+, f^-, f^-, x_2] = 4x_2$ implies $x_2 = 0$, hence $y = 0$. ■

Proposition 5.4. Let $L = \bigoplus_{\gamma\in\Gamma} L^\gamma$ be a Γ-graded Lie algebra and let B be a Γ-graded abelian inner ideal of L. Suppose further that $\mathcal{E} = (e_\alpha)_{\alpha\in A}$ is a toral family of homogeneous idempotents such that all $e_\alpha^+ \in B$. We thus have a $\mathbb{Z}^A$-grading $L = \bigoplus_\omega L_\omega$ as defined in (5.3) which is compatible with the given Γ-grading.

(a) Put $B_\omega = B \cap L_\omega$ for $w \in \mathbb{Z}^A$. Then $B = \oplus_\omega B_\omega$, where each B_ω is a Γ-graded submodule. Moreover, $B_\omega \neq 0$ only when $\omega(\alpha) \geq 0$ for all $\alpha \in A$, and if $\omega(\alpha) = 2$ for some $\alpha \in A$ then $B_\omega = L_\omega$.

(b) Let $K = \mathrm{Ker}_L B$ and put $K_\omega = K \cap L_\omega$ for $w \in \mathbb{Z}^A$. Then $K = \oplus_\omega K_\omega$, where each K_ω is a Γ-graded submodule and $K_\omega = 0$ if $\omega(\alpha) = -2$ for some $\alpha \in A$.

(c) Let $V = (B, L/K)$, and put $g_\alpha = \overline{e_\alpha} = (e_\alpha^+, \overline{e_\alpha^-}) \in V$. Then $\overline{\mathcal{E}} = \mathcal{G} = (g_\alpha)_{\alpha\in A}$ is a compatible family of homogeneous idempotents in V whose joint Peirce spaces are

$$V_\omega(\mathcal{G}) = (B_\omega, L_{-\omega}/K_{-\omega}).$$

Hence $V = \oplus_\omega (B_\omega, L_{-\omega}/K_{-\omega})$.

(d) Let $f = (f^+, f^-) \in V_\omega^\gamma(\mathcal{G}) = \big((V_\omega^+)^\gamma, (V_\omega^-)^{-\gamma}\big)$ be a homogeneous idempotent of V. Then there exists a homogeneous idempotent $e \in (L_\omega^\gamma, L_{-\omega}^{-\gamma})$ such that $(e^+, \overline{e^-}) = f$. The extended family $\mathcal{E} \cup \{e\}$ is again toral. Moreover, if $\mathcal{E}$ and $\mathcal{G} \cup \{f\}$ are Peirce-compatible families, then so is $\mathcal{E} \cup \{e\}$. □

Proof.

(a) We have $[h_\alpha, B] = [[e_\alpha^+, e_\alpha^-], B] = [e_\alpha^+, [e_\alpha^-, B]] \subset B$ since B is an abelian inner ideal of L. This implies $B = \oplus_\omega B_\omega$ and that each B_ω is homogeneous.

If $\omega(\alpha) < 0$ for some $\alpha \in A$, for $b \in B_\omega$ we have $\omega(\alpha)b = [h_\alpha, b] = [e_\alpha^+, [e_\alpha^-, b]] = 0$, since $[e_\alpha^-, b] \in L(h_\alpha)_{\omega(\alpha)-2} = 0$ because $|\omega(\alpha) - 2| \geq 3$.

If $\omega(\alpha) = 2$, then $L_\omega \subset L_2(h_\alpha) = [e_\alpha^+, e_\alpha^+, L] \subset B$.

(b) follows from $[K, L, B] \subset K$, using (3.1) and Lemma 5.3 (b).

(c) It is immediate from the definition of the Jordan triple product of V that $\overline{\mathcal{E}}$ is a family of homogeneous idempotents. Indeed, we have $\{g_\alpha^+, g_\alpha^-, b\} = [[e_\alpha^+, e_\alpha^-], b] = [h_\alpha, b]$ and, $\{g_\alpha^+, g_\alpha^-, \overline{x}\} = -\overline{[h_\alpha, x]}$ for $b \in B$ and $x \in L$. These formulas also show that the left multiplication operators $D(e_\alpha^\sigma, e_\alpha^{-\sigma})$ in V are given by $D(g_\alpha^+, g_\alpha^-) = \operatorname{ad} h_\alpha$ on $B = V^+$, and $D(g_\alpha^+, g_\alpha^-) = -\mathrm{can} \circ \operatorname{ad} h_\alpha$ on V^- for can: $L \to L/K$ the canonical map. It follows that $\mathcal{G}$ is a compatible family of idempotents of V. For $\omega \in \mathbb{Z}^A$ and $b \in B$ we have $b \in B_\omega \Leftrightarrow [h_\alpha, b] = \omega(\alpha)b$ for all $\alpha \in A \Leftrightarrow b \in V_{\omega(\alpha)}^+(g_\alpha)$ for all $\alpha \in A$. Also for $x = \sum_{\nu\in\mathbb{Z}^A} x_\nu$, $x_\nu \in L_\nu$, we get $\{g_\alpha^-, g_\alpha^+, \overline{x}\} = -\sum_\nu \nu(\alpha)\overline{x}_\nu$. From this it easily follows that $V_\omega^-(\mathcal{G}) = L_{-\omega}/K_{-\omega}$.

(d) Put $e^+ = f^+$. We have $(\operatorname{ad} e^+)^3 L = [e^+, (\operatorname{ad} e^+)^2 L] \subset [e^+, B] = 0$ since B is an abelian inner ideal. Let $u \in L_{-\omega}^{-\gamma}(\mathcal{E})$ such that $\overline{u} = f^- \in V_\omega(\mathcal{G})^-$. Then $[[e^+, u], e^+] = [[e^+, f^-], e^+] = 2e^+$. By Proposition 5.2 there exists $v \in L_{-\omega}^{-\gamma}(\mathcal{E}) \cap \operatorname{Ker} \operatorname{ad} e^+$ such that $(e^+, [e^+, e^-], e^-)$, $e^- = u - v$, is an $\mathfrak{sl}_2$-triple with $(\operatorname{ad} e^-)^3 = 0$, i.e., $e = (e^+, e^-)$ is a homogeneous idempotent of L. Since $h_e = [e^+, e^-] \in L_0^0$, the extended family is again toral. Also,

$[[e^-, u], e^+] \in \mathrm{Ker}_L B$, so

$$\begin{aligned} 2f^- &= [[f^-, f^+], f^-] = \overline{[[u, e^+], u]} = \overline{[[e^-, e^+], u]} = \overline{[[e^-, u], e^+] + [e^-, [e^+, u]]} \\ &= \overline{[[e^-, e^+], e^-] = -[h_e, e^-] = 2e^-}. \end{aligned}$$

Hence $f^- = \overline{e^-}$, and e is indeed a lift of f.

By construction $e \in L^{\gamma}_{\omega}(\mathcal{E})$. For the second part of (d) it therefore remains to prove that (e^+_α, e^-_α), $\alpha \in A$, lies in the Peirce space of e, i.e., $[h, e^\sigma_\alpha] = \sigma\mu e^\sigma_\alpha$ for $h = [e^+, e^-]$, $\sigma = \pm$, and some $\mu \in \{0, \pm1, \pm2\}$. But we know that there exists $\mu \in \{0, \pm1, \pm2\}$ such that $\{f^\sigma, f^{-\sigma}, g^\sigma_\alpha\} = \mu g^\sigma_\alpha$, so in particular $[h, e^+_\alpha] = [[e^+, e^-], e^+_\alpha] = [[f^+, f^-], g^+_\alpha] = \mu g^+_\alpha$, while $\overline{[h, e^-_\alpha]} = \overline{-[[e^-, e^+], e^-_\alpha]} = -\{f^-, f^+, g^-_\alpha\} = -\mu\overline{e^-_\alpha}$, so $[h, e^-_\alpha] + \mu e^-_\alpha \in K_\omega$. Since $\mathcal{E}$ is Peirce-compatible, $e^-_\alpha \in L_\omega$ for some $\omega \in \mathbb{Z}^A$, whence $K_\omega = 0$ by (b). Therefore $[h, e^-_\alpha] = -\mu e^-_\alpha$. ■

5.2 Weight-graded Lie algebras

Let R be a root system. A $\mathcal{P}(R)$-graded Lie algebra $L = \bigoplus_{\omega \in \mathcal{P}(R)} L_\omega$ is called an *R-weight-graded Lie algebra* [27] if it has the following properties:

(i) For every $\alpha \in R^\times = R \setminus \{0\}$ there exists a non-zero pair $(e_\alpha, f_\alpha) \in L_\alpha \times L_{-\alpha}$ such that $h_\alpha = [e_\alpha, f_\alpha]$ acts on L_ω by $[h_\alpha, x_\omega] = \langle \omega, \alpha^\vee \rangle x_\omega$ where $x_\omega \in L_\omega$.

(ii) $L_0 = \sum_{0 \neq \omega \in \mathcal{P}(R)} [L_\omega, L_{-\omega}]$.

(iii) For all $\sigma, \tau \in \mathrm{supp}\, L = \{\omega \in \mathcal{P}(R) : L_\omega \neq 0\}$ there exists $\alpha \in R$ such that $\langle \sigma - \tau, \alpha^\vee \rangle . 1_\Phi \in \Phi^\times$.

In this case, we will call $\mathcal{S} = (e_\alpha, f_\alpha : \alpha \in R^\times)$ a *splitting family* and simply write $\mathcal{S} = (e_\alpha : \alpha \in R^\times)$ in case $\mathcal{S}$ is *normalized* in the sense that $f_\alpha = e_{-\alpha}$. An *R-graded Lie algebra* as defined in [26] is an R-weight-graded Lie algebra with $\mathrm{supp}\, L = R$ a reduced root system.

We note that (iii) is of course automatic if Φ is a field of characteristic 0. Also, if L is R-graded, (iii) just means $2, 3 \in \Phi^\times$ and hence is always fulfilled under our assumption on Φ. For R a finite root system and Φ a field, the notion of an R-graded Lie algebra has been introduced and studied by Berman-Moody in case R is simply-laced and $\neq \mathrm{A}_1$ [2], and by Benkart-Zelmanov in the remaining cases [5].

If L only satisfies (i) and (iii), it is easily checked that then

$$L_c = \Big(\sum_{0 \neq \omega} [L_\omega, L_{-\omega}] \Big) \oplus \Big(\bigoplus_{0 \neq \omega} L_\omega \Big) \tag{5.4}$$

is an ideal of L, called the *core*, which is R-weight-graded.

The next theorem uses the concept of standard grids in Jordan pairs for which the reader is referred to [[23]; 3.5] or [[25]; 1.7]. We note that every covering grid can be changed to a covering standard grid with the same Peirce spaces and the same associated 3-graded root system. We also recall our basic assumption for this section: All algebraic structures are defined over Φ in which $i \cdot 1_\Phi$, $i = 2, 3, 5$, is invertible.

Theorem 5.5. Let B be an abelian inner ideal in a Lie algebra L, and suppose that the subquotient $V = (B, L/\mathrm{Ker}_L B)$ is covered by a standard grid $\mathcal{G}$ with associated 3-graded root system (R, R_1), hence $\mathcal{G} = (g_\alpha)_{\alpha \in R_1}$ for idempotents $g_\alpha = (g_\alpha^+, g_\alpha^-)$ in V. Let $\mathcal{E} = (e_\alpha)_{\alpha \in R_1}$ be a toral family of Peirce-compatible idempotents in L such that $\overline{e_\alpha} = (e_\alpha^+, \overline{e_\alpha^-}) = g_\alpha$ for all $\alpha \in R_1$. We put $h_\alpha = [e_\alpha^+, e_\alpha^-]$ and denote by L_ω, $\omega \in \mathbb{Z}^{R_1}$, the joint eigenspaces of $(h_\alpha)_{\alpha \in R_1}$:

$$L_\omega = \{x \in L : [h_\alpha, x] = \omega(\alpha)x \text{ for all } \alpha \in R_1\}. \tag{5.5}$$

We let $\mathrm{supp}\, L = \{\omega : L_\omega \neq 0\}$.

(a) Every $\omega \in \mathrm{supp}\, L$ has a unique extension to a weight of R, also denoted ω, such that $\omega(\alpha) = \langle \omega, \alpha^\vee \rangle$ for all $\alpha \in R_1$. Moreover, putting $L_\omega = 0$ for $\omega \in \mathcal{P}(R) \setminus \mathrm{supp}\, L$, the decomposition $L = \bigoplus_{\omega \in \mathcal{P}(R)} L_\omega$ is a grading by the abelian group $\mathcal{P}(R)$.

(b) Every $\omega \in \mathrm{supp}\, L$ has the property that $\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle \in \{0, \pm 1 \pm 2\}$ for every finite orthogonal system $\mathcal{O} \subset R_1$. Moreover, for $\sigma = \pm$ we have $\omega \in R_{\sigma 1}$ if and only if there exists a finite orthogonal system $\mathcal{O} \subset R_1$ such that $\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle = \sigma 2$.

(c) $B = \bigoplus_{\alpha \in R_1} L_\omega$ and $\mathrm{Ker}_L B = \bigoplus_{\omega \notin R_{-1}} L_\omega$.

(d) For $0 \neq \mu \in R_0$, written as $\mu = \alpha - \beta$ with $\alpha, \beta \in R_1$, the element $e_\mu = [e_\alpha^+, e_\beta^-]$ is well-defined up to sign. Moreover, L satisfies the conditions (i) and (iii) of 5.2 with respect to the family $\mathcal{S} = (e_\alpha, : \alpha \in R^\times)$ where $e_\alpha = e_\alpha^+$, $e_{-\alpha} = e_\alpha^-$ for $\alpha \in R_1$ and e_μ as defined above for some chosen decomposition $0 \neq \mu = \alpha - \beta \in R_0$. Hence the core

$$L_c = \left(\textstyle\sum_{0 \neq \omega} [L_\omega, L_{-\omega}]\right) \oplus \left(\textstyle\bigoplus_{0 \neq \omega} L_\omega\right)$$

of L, cf. (5.4), is an R-weight-graded ideal of L.

(e) Let $\mathfrak{h} = \sum_{\alpha \in R_1} \Phi h_\alpha \subset L_0$. Then $\mathfrak{h}$ is an abelian subalgebra of L, and

$$\mathfrak{g} = \left(\textstyle\bigoplus_{\alpha \in R_1} \Phi e_\alpha^+\right) \oplus \left(\mathcal{H} \oplus \textstyle\sum_{0 \neq \mu \in R_0} \Phi e_\mu\right) \oplus \left(\textstyle\bigoplus_{\alpha \in R_{-1}} \Phi e_\alpha^-\right)$$

is a subalgebra of L which is R-graded and hence in particular 3-graded. If Φ is a field of characteristic 0, then $\mathfrak{g}$ is the Tits-Kantor-Koecher algebra of the Jordan pair spanned by $\mathcal{G}$. In particular, if Φ is a field of characteristic 0 and R is finite then $\mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$. □

The proof of the theorem will be given in 5.3. In the Lemmas 5.6–5.9 we will establish some additional results on the structure of L which are of independent interest. Throughout the assumptions of Theorem 5.5 are assumed to hold, except that we do not assume (nor use) that $\mathcal{G}$ covers V.

Lemma 5.6. Let $\alpha, \beta, \gamma \in R_1$. Then for $\sigma = \pm$ we have $[\,[e^\sigma_\alpha, e^{-\sigma}_\alpha], e^\sigma_\beta] = \langle \beta, \alpha^\vee \rangle e^\sigma_\beta = [\,[e^\sigma_\beta, e^{-\sigma}_\alpha], e^\sigma_\alpha]$, while for $\alpha \neq \beta \neq \gamma$ and $\alpha - \beta + \gamma = \delta \in R_1$ there exists $\mu \in \{\pm 1, 2\}$ such that $[\,[e^\sigma_\alpha, e^{-\sigma}_\beta], e^\sigma_\gamma] = \mu e^\sigma_\delta = [\,[e^\sigma_\gamma, e^{-\sigma}_\beta], e^\sigma_\alpha]$ for $\sigma = \pm$, where μ is determined by the corresponding equation for $\mathcal{G}$, i.e., $\{g^\sigma_\alpha, g^{-\sigma}_\beta, g^\sigma_\delta\} = \mu g^\sigma_\delta$. □

Proof. By [[23]; 3.5] all equations hold for g^σ in place of e^σ. Since $[[e^+_\alpha, e^-_\beta], e^+_\gamma] = \{g^+_\alpha, g^-_\beta, g^+_\gamma\}$, the claim holds for $\sigma = +$. For $\sigma = -$ we get $\overline{[\,[e^-_\alpha, e^+_\beta], e^-_\gamma]} = \{g^-_\alpha, g^+_\beta, g^+_\gamma\} = \nu g^-_\delta$, where $\nu = \langle \beta, \alpha^\vee \rangle$ for the first formula and $\nu = \mu$ for the second. By grading properties, $[[e^-_\alpha, e^+_\beta], e^-_\gamma], e^-_\delta \in L_\omega$ for a suitable ω, where $g_\delta \in V_{-\omega} = (L_{-\omega}, L_\omega)$ by Proposition 5.4 (c). Since $\omega(\delta) = -2$, we have $K_\delta = 0$ by Proposition 5.4 (b), whence $[[e^-_\alpha, e^+_\beta], e^-_\gamma] = \nu e^-_\delta$. We also get $[[e^-_\alpha, e^+_\beta], e^-_\gamma] - [[e^-_\gamma, e^+_\beta], e^-_\alpha] \in K_\delta = 0$. ■

Lemma 5.7. For $\alpha, \beta \in R_1$ with $\alpha \perp \beta$ we have $[e^+_\alpha, e^-_\beta] = 0$ and $[\,[e^\sigma_\alpha, e^{-\sigma}_\beta], e^\sigma_\gamma] = [[e^\sigma_\gamma, e^{-\sigma}_\beta], e^\sigma_\alpha]$ $= 0$ for all $\gamma \in R_1$. □

Proof. It is immediate from the definitions that $[h_\beta, e^+_\alpha] = [[e^+_\beta, e^-_\beta], e^+_\alpha] = \langle \alpha, \beta^\vee \rangle e^+_\alpha = 0$ and $[h_\beta, e^-_\beta] = -2e^-_\beta$. Hence $[h_\beta, [e^+_\alpha, e^-_\beta]] = -2[e^+_\alpha, e^-_\beta] \in L_{-2}(h_\beta)$. Thus $[e^+_\alpha, e^-_\beta] \in L_\omega$ where $\omega(\beta) = -2$. But $[e^+_\alpha, e^-_\beta] \in [B, L] \subset K$ by Lemma 3.2 (c). Since $K_\omega = 0$ by Proposition 5.4 (b), we have $[e^+_\alpha, e^-_\beta] = 0$. The second equation is clear for $\sigma = +$, since it holds in V, cf. Lemma 5.6. For $\sigma = -$, we have $[\,[e^-_\alpha, e^+_\beta], e^-_\gamma] = 0$ since $[e^-_\alpha, e^+_\beta] = 0$ by what we just proved. Moreover, $[\,[e^-_\gamma, e^+_\beta], e^-_\alpha] \in L_{-(\langle \gamma, \alpha^\vee \rangle + 2)}(h_\alpha)$ by Theorem 5.5 (a), whence $[\,[e^-_\gamma, e^+_\beta], e^-_\alpha] = 0$ when $\langle \gamma, \alpha^\vee \rangle > 0$, while $[[e^-_\gamma, e^+_\beta], e^-_\alpha] \in L_{-2}(h_\alpha) \cap K$ for $\gamma \perp \alpha$, which again implies $[[e^-_\gamma, e^+_\beta], e^-_\alpha] = 0$. ■

Lemma 5.8.

(a) Let $(\alpha; \beta, \gamma) \subset R_1$ be a triangle. Then $h_\alpha = h_\beta + h_\gamma$ and $[e^+_\alpha, e^-_\beta] = [e^+_\gamma, e^-_\alpha]$.

(b) Let $(\alpha, \beta, \gamma, \delta) \subset R_1$ be a quadrangle. Then $h_\alpha + h_\gamma = h_\beta + h_\delta$. Moreover, $[e^+_\beta, e^-_\alpha] = \epsilon [e^+_\gamma, e^-_\delta]$ where the sign $\epsilon \in \{\pm\}$ is determined from $\{g^\sigma_\alpha, g^{-\sigma}_\beta, g^\sigma_\gamma\} = \epsilon g^\sigma_\delta$.

(c) Let $(\alpha; \beta, \gamma, \delta) \subset R_1$ be a diamond. Then $2h_\alpha + h_\gamma = h_\beta + h_\delta$ and $[e^+_\beta, e^-_\alpha] = [e^+_\gamma, e^-_\delta]$. □

Proof. In Lemma 5.6 and Lemma 5.7 we have established all necessary equations so that the proof of [[23]; Lemma 2.2] works in our more general situation. Details will be left to the reader. ■

Lemma 5.9. Let $\mathcal{O} \subset R_1$ be a finite orthogonal system and define $e^\sigma_{\mathcal{O}} = \sum_{\alpha \in \mathcal{O}} e^\sigma_\alpha$. Then $e_{\mathcal{O}} = (e^+_{\mathcal{O}}, e^-_{\mathcal{O}})$ is an idempotent of L with $[e^+_{\mathcal{O}}, e^-_{\mathcal{O}}] = \sum_{\alpha \in \mathcal{O}} h_\alpha$ and $\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle \in \{0, \pm 1, \pm 2\}$ for any $\omega \in \operatorname{supp} L$. □

Proof. It is immediate from Lemma 5.6 and Lemma 5.7 that $h_{\mathcal{O}} = [e^+_{\mathcal{O}}, e^-_{\mathcal{O}}] = \sum_{\alpha \in \mathcal{O}} [e^+_\alpha, e^-_\alpha]$ and that $[h_{\mathcal{O}}, e^\sigma_{\mathcal{O}}] = \sigma 2 e^\sigma_{\mathcal{O}}$. Since $(\operatorname{ad} e^+)^3 = 0$ it then follows that $(e^+_{\mathcal{O}}, e^-_{\mathcal{O}})$ is an idempotent. For $0 \neq x \in L_\omega$ we have $[h_{\mathcal{O}}, x] = \sum_{\alpha \in \mathcal{O}} [h_\alpha, x] = (\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle)\, x$. Now, if $|\sum_{\alpha \in \mathcal{O}} \langle \omega, \alpha^\vee \rangle| \geq 3$ there exists a subsystem $\mathcal{O}' \subset \mathcal{O}$ of cardinality 2 or 3 such that $\sum_{\alpha \in \mathcal{O}'} \langle \omega, \alpha^\vee \rangle = \{\pm 3, \pm 4\}$. Hence $[h', x] = \mu x$ for $\mu \in \{\pm 3, \pm 4\}$. However, since $f = (\sum_{\alpha \in \mathcal{O}'} e^+_\alpha, \sum_{\alpha \in \mathcal{O}'} e^-_\alpha)$ is an idempotent of $\mathcal{L}$ with $[f^+, f^-] = h'$, the eigenvalues λ_i of $\operatorname{ad} h'$ lie in $\{0, \pm 1, \pm 2\}$. Since for $\mu \in \{\pm 3, \pm 4\}$ we have $\lambda_i - \mu \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\} \cdot 1_\Phi \subset \Phi^\times$, the equation $[h', x] = \mu x$ implies $x = 0$, contradiction. ■

5.3 Proof of Theorem 5.5

(a) For the proof of Theorem 6.3 below we point out that we will not use in the proof of (a) that $\mathcal{G}$ covers V.

By Proposition 4.1 (b) it suffices to check that for $\omega \in \operatorname{supp}(L)$ we have:

(i) $\omega(\alpha) = \omega(\beta) + \omega(\gamma)$ for any triangle $(\alpha; \beta, \gamma) \subset R_1$, and

(ii) $\omega(\alpha) + \omega(\gamma) = \omega(\beta) + \omega(\delta)$ for any quadrangle $(\alpha, \beta, \gamma, \delta) \subset R_1$.

Let $0 \neq x \in L_\omega$, and let $(\alpha; \beta, \gamma) \subset R_1$ be a triangle. Thus, by Lemma 5.8 (a), $\omega(\alpha) x = [h_\alpha, x] = [h_\beta + h_\gamma, x] = (\omega(\beta) + \omega(\delta)) x$, i.e., $(\omega(\alpha) - \omega(\beta) - \omega(\delta)) x = 0$. Since $|\omega(\mu)| \leq 2$ for any $\mu \in R_1$, we have that $|\omega(\alpha) - \omega(\beta) - \omega(\delta)| \leq 6$. Hence, if $\omega(\alpha) - \omega(\beta) - \omega(\delta) \neq 0$, then $\omega(\alpha) - \omega(\beta) - \omega(\delta)$ is invertible in Φ, so $x = 0$ follows. Thus (i) holds. Similarly, if $(\alpha, \beta, \gamma, \delta) \subset R_1$ is a quadrangle, we obtain from Lemma 5.8 (b) that $(\omega(\alpha) + \omega(\gamma) - \omega(\beta) - \omega(\delta)) x = 0$. We apply Lemma 5.9 to the orthogonal systems (α, γ) and (β, δ) and we get $|\omega(\alpha) + \omega(\gamma) - (\omega(\beta) + \omega(\delta))| \leq |\omega(\alpha) + \omega(\gamma)| + |\omega(\beta) + \omega(\delta)| \leq 2 + 2 = 4$, hence if

$\omega(\alpha)+\omega(\gamma)-\omega(\beta)-\omega(\delta)\neq 0$, then $\omega(\alpha)+\omega(\gamma)-\omega(\beta)-\omega(\delta)$ is invertible in Φ and $x=0$ follows. Because $L_\omega\neq 0$ we obtain $\omega(\alpha)+\omega(\gamma)-\omega(\beta)-\omega(\delta)=0$, i.e., (ii). Thus $\omega\in\operatorname{supp}L$ uniquely extends to a weight, also denoted by ω. That $L=\bigoplus_{\omega\in\mathcal{P}(R)}L_\omega$ is a $\mathcal{P}(R)$-grading is now immediate from (5.5).

(b) If $\omega(\alpha)=2$ for some $\alpha\in R_1$, then $L_\omega\subset L_2(h_\alpha)\subset B$ by Proposition 5.4 (a), in particular $\bigoplus_{\alpha\in R_1}L_\alpha\subset B$. Conversely, if $B_\omega\neq 0$ then, by Proposition 5.4 (c), $B_\omega=V_\omega^+(\mathcal{G})$ is a Peirce space of V with respect to $\mathcal{G}$. Since $\mathcal{G}$ covers V, we get $\omega=\alpha$ for some $\alpha\in R_1$. Thus $B=\bigoplus_{\alpha\in R_1}L_\alpha$.

We know from Proposition 5.4 (b) that $K=\bigoplus_\omega K_\omega$ where $K_\omega=K\cap L_\omega$, and $K_\omega=0$ if $\omega(\alpha)=-2$ for some $\alpha\in R_1$, whence $K=\bigoplus_{\omega\notin R_{-1}}K_\omega\subset\bigoplus_{\omega\notin R_1}L_\omega$. Conversely, by Proposition 5.4 (c) any $L_\omega/K_\omega\subset V^-$ is a Peirce space of $\mathcal{G}$. Since the Peirce spaces of $\mathcal{G}$ in V^- are V_α^-, $\alpha\in R_1$, we have $L_\omega/K_\omega=0$ if $\omega\notin R_{-1}$, i.e., $L_\omega=K_\omega$ for those ω.

(c) The first part of (b) was proven in Lemma 5.9. For the second part, the condition is obviously necessary: If $\omega=\alpha\in R_{\sigma 1}$, $\sigma=\pm$, then $\sigma\alpha$ is an orthogonal system in R_1 with $\langle\omega,(\sigma\alpha)^\vee\rangle=\sigma\langle\alpha,\alpha^\vee\rangle=2\sigma$. Conversely, if $\mathcal{O}\subset R_1$ is an orthogonal system with $\sum_{\alpha\in\mathcal{O}}\langle\omega,\alpha^\vee\rangle=2\sigma$, let $e_\mathcal{O}=(e_\mathcal{O}^+,e_\mathcal{O}^-)$ be the idempotent of Lemma 5.9, and put $h_\mathcal{O}=[e_\mathcal{O}^+,e_\mathcal{O}^-]$. If $\sigma=+$ then $L_\omega\subset L_2(h_\mathcal{O})=[[e_\mathcal{O}^+,e_\mathcal{O}^+],L]\subset B$, so $L_\omega\subset V^+$ is a Peirce space with respect to $\mathcal{G}$ and therefore of the form $L_\omega=L_\beta$ for some $\beta\in R_1$. If $\sigma=-$ then $L_\omega\subset L_{-2}(h_\mathcal{O})$. Since $e_\mathcal{O}^+\in B$, Lemma 5.3 (b) shows $K_\omega=0$, whence $L_\omega\subset V^-$ is a Peirce space with respect to $\mathcal{G}$, and therefore of the form L_β for some $\beta\in R_{-1}$.

(d) That e_μ, $0\neq\mu\in R_0$, is well defined can be proven in the same way as [[25]; Lemma 2.4] by using Lemma 5.8 and [[19]; Prop. 18.9] in place of the results quoted in the proof of [[25]; Lemma 2.4].

Condition (i) of 5.2 for $\alpha\in R_1$ follows from (5.5) since $[e_\alpha,f_\alpha]=h_\alpha$ as defined in the theorem. It then also holds for $\alpha\in R_{-1}$. For $0\neq\mu=\alpha-\beta\in R_0$ we have $[e_\mu,e_{-\mu}]=[[e_\alpha^+e_\beta^-],[e_\beta^+e_\alpha^-]]=[[[e_\alpha^+e_\beta^-]e_\beta^+],e_\alpha^-]-[e_\beta^+,[[e_\beta^-e_\alpha^+]e_\alpha^-]]=\langle\alpha,\beta^\vee\rangle h_\alpha-\langle\beta,\alpha^\vee\rangle h_\beta$ by Lemma 5.6. Since $\mu^\vee=\langle\alpha,\beta^\vee\rangle\alpha^\vee-\langle\beta,\alpha^\vee\rangle\beta^\vee$ by [[19]; A.4], condition (i) of 5.2 also holds for $\mu\in R_0$. Condition (iii) of 5.2 holds because of (b) and our assumption on Φ. As already mentioned in 5.2, the core of L is then R-weight-graded.

(e) $\mathfrak{h}$ is abelian by compatibility of $\mathcal{E}$. To check that $\mathfrak{g}$ is a subalgebra of L, we put $\mathfrak{g}_\epsilon=\mathfrak{g}\cap L_\epsilon$ and thus have $\mathfrak{g}=\bigoplus_{\epsilon\in R}\mathfrak{g}_\epsilon$ with

$$\mathfrak{g}_\epsilon=\begin{cases}\mathfrak{h} & \text{for } \epsilon=0,\\ \Phi e_\epsilon & \text{for } 0\neq\epsilon\in R_0,\\ \Phi e_\alpha^\pm & \text{for } \pm\alpha\in R_{\pm 1}.\end{cases}$$

Clearly $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{g}$. In the following we will consider the products $[\mathfrak{g}_\epsilon, \mathfrak{g}_\nu]$ for $0 \neq \epsilon, \nu \in R$ and distinguish the cases (1) $\epsilon, \nu \in R_1$, (2) $\epsilon, \nu \in R_{-1}$, (3) $\epsilon \in R_1, \nu \in R_{-1}$, (4) $0 \neq \epsilon \in R_0$, $\nu \in R_1$, (5) $0 \neq \epsilon \in R_0, \nu \in R_{-1}$ and (6) $0 \neq \epsilon, \nu \in R_0$

(1) Let $\epsilon = \alpha \in R_1$ and $\nu = \beta \in R_1$: We have $[e^+_\alpha, e^+_\beta] = 0$ since B is abelian.

(2) Let $\epsilon = -\alpha \in R_{-1}$ and $\nu = -\beta \in R_1$: If $\alpha \not\perp \beta$ then $[e^-_\alpha, e^-_\beta] \in L_{-\alpha-\beta} = 0$ because of $\langle \alpha+\beta, \alpha^\vee \rangle \geq 3$ and (c). If $\alpha \perp \beta$ the assumption $L_{-\alpha-\beta} \neq 0$ together with $-\langle \alpha+\beta, \alpha^\vee\rangle = -2$ implies the contradiction $\alpha + \beta \in R_1$, whence again $[e^-_\alpha, e^-_\beta] = 0$.

(3) Let $\epsilon = \alpha \in R_1$ and $\nu = -\beta \in R_{-1}$: If $\alpha = \beta$ then $[e^+_\alpha, e^-_\beta] = h_\alpha \in \mathfrak{h}$. If $\alpha \neq \beta$ but $\alpha \not\perp \beta$ then $0 \neq \mu = \alpha - \beta \in R_0$ and $[e^+_\alpha, e^-_\beta] = \pm e_\mu \in \mathfrak{g}$. Finally, if $\alpha \perp \beta$ then $[e^+_\alpha, e^-_\beta] = 0$ by Lemma 5.7.

(4) Let $0 \neq \epsilon \in R_0$ and $\nu = \gamma \in R_1$: We can write ϵ in the form $\epsilon = \alpha - \beta$ for suitable $\alpha, \beta \in R_1$, $\alpha \not\perp \beta$ such that $[e_\mu, e^+_\gamma] = [[e^+_\alpha, e^-_\beta], e^+_\gamma] = \{g^+_\alpha, g^-_\beta, g^+_\gamma\}$. By [[23]; 3.5] this element is zero if $\alpha - \beta + \gamma = \delta \notin R_1$, and lies in Φe^+_δ if $\delta \in R_1$, cf. Lemma 5.6.

(5) Let $0 \neq \epsilon \in R_0$ and $\nu = -\gamma \in R_1$: As in case (4) we let $\mu = \alpha - \beta$ so that $e_\mu = [e^+_\alpha, e^-_\mu]$. Since then $[e_\mu, e^-_\gamma] = -[[e^-_\beta, e^+_\alpha], e^-_\gamma]$ we are again done by Lemma 5.6 in case $\omega = \beta - \alpha + \gamma \in R_1$. Let us therefore assume $\omega \notin R_1$. We claim than then $[[e^-_\beta, e^+_\alpha], e^-_\gamma] = 0$. Because $\{g^-_\beta, g^+_\alpha, g^-_\gamma\} = 0$ we get at least $[[e^-_\beta, e^+_\alpha], e^-_\gamma] \in K_{-\omega}$. We can of course assume $K_{-\omega} \neq 0$. By Lemma 5.8 and Proposition 5.4 (c) we then have $\langle \omega, \delta^\vee \rangle \leq 1$ for all $\delta \in R_1$, in particular for $\delta = \beta$ and $\delta = \gamma$ we get $1 + \langle \gamma, \beta^\vee \rangle \leq \langle \lambda, \beta^\vee \rangle$ and $1 + \langle \beta, \gamma^\vee \rangle \leq \langle \alpha, \gamma^\vee \rangle$. By Lemma 5.7 we can also assume $\beta \not\perp \alpha \not\perp \gamma$. The inequalities above together with $\langle R_1, R_1^\vee \rangle \leq 2$ then imply $\beta \vdash \alpha \dashv \gamma \top \beta$. But then $(\alpha; \beta, \omega, \gamma)$ is a diamond by [[19]; 18.4] with $\omega \in R_1$. This contradiction proves $[e_\mu, e_\nu] = 0$ in this case.

(6) Finally, let $0 \neq \epsilon, \nu \in R_0$: We write again ϵ as in (4) and can assume that $\nu = \gamma - \delta$ for suitable $\gamma, \delta \in R_1$. Then $[e_\mu, e_\nu] = [e_\mu, [e^+_\gamma, e^-_\delta]] = [[e_\mu, e^+_\gamma], e^-_\delta] + [e^+_\gamma, [e_\mu, e^-_\delta]] \in \mathfrak{g}$ by what we have already proven. This finishes the proof that $\mathfrak{g}$ is a subalgebra.

That $\mathfrak{g}$ is R-graded is now immediate from (d) and the definition of an R-graded algebra. Since R is a 3-graded root system, $\mathfrak{g}$ is a 3-graded Lie algebra. The last statements then follow from [[25]; Theorem 3.3 and Theorem 3.4].

6 Consequences and Examples

In this section we will draw some consequences of Theorem 5.5. As in the previous section we assume that all Lie algebras and Jordan pairs are defined over a ring of scalars Φ with $\mu 1_\Phi \in \Phi^\times$ for $\mu = 2, 3, 5$.

Following [11] we will say that an abelian inner ideal B of a Lie algebra L is *complemented by an abelian inner ideal* if there exists an abelian inner ideal C of L such that $L = B \oplus \operatorname{Ker}_L C = C \oplus \operatorname{Ker}_L B$.

Theorem 6.1. Let L be a Γ-graded Lie algebra, and let B be a graded abelian inner ideal such that the subquotient $V = (B, L/\operatorname{Ker}_L B)$ is covered by a finite grid $\mathcal{G}$ of homogeneous idempotents. Let R be the finite 3-graded root system associated to $\mathcal{G}$.

Then the assumptions of Theorem 5.5 are fulfilled. In particular, the $\mathcal{P}(R)$-grading $L = \bigoplus_{\omega \in \mathcal{P}(R)} L_\omega$ of L is compatible with the given Γ-grading of L. Moreover:

(a) L has a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ which is compatible with the Γ-grading of L and satisfies

$$L_n = B, \quad \operatorname{Ker}_L B = L_{-n+1} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n. \tag{6.1}$$

If $\mathcal{G}$ is a connected grid, then n in (6.1) can be taken as the Coxeter number of R.

(b) $C = L_{-n}$ is also a graded abelian inner ideal of L with $\operatorname{Ker}_L C = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots L_{n-1}$. In particular, B is complemented by C. □

Proof. If V is covered by a finite grid of homogeneous idempotents, it follows from [[23]; Theorem 3.7] that V is also covered by a finite standard grid of homogeneous idempotents, say $\mathcal{G} \subset V$. By repeated application of Proposition 5.4 (d) we can construct a finite Peirce-compatible family $\mathcal{E}$ of homogeneous idempotents of L such that the assumptions of Theorem 5.5 are fulfilled. In particular, $L = \bigoplus_{\omega \in \mathcal{P}(R)} L_\omega$ is graded by $\mathcal{P}(R)$. Since all h_α, $\alpha \in R_1$, lie in L_0^0 this $\mathcal{P}(R)$-grading is compatible with the given Γ-grading.

(a) Let $\varphi\colon \mathcal{P}(R) \to \mathbb{Z}$ be the homomorphism of Proposition 4.3. We regrade L via φ, i.e., we define $L_i = \bigoplus_{\varphi(\omega)=i} L_\omega$ for $i \in \mathbb{Z}$. The remaining statements of (a) then follow from Theorem 5.5 (c) and Proposition 4.3, keeping in mind for the last part that $\mathcal{G}$ is connected if and only if R is irreducible [[23]; Theorem 3.4].

(b) That also C is an abelian inner ideal is obvious, cf. Lemma 2.3. We have $L_{-n} = \bigoplus_{\omega \in R_{-1}} L_\omega$, and hence C also fulfills the assumptions of Theorem 5.5 with

V replaced by V^{op} and $+/-$ exchanged in $\mathcal{G}$ and $\mathcal{E}$. Then Theorem 5.5 (c) shows that $\mathrm{Ker}_L\, C$ is as claimed in the theorem. It then follows that $L = B \oplus \mathrm{Ker}_L\, C = C \oplus \mathrm{Ker}_L\, B$, i.e., B is complemented by C. ■

Corollary 6.2. Let L be nondegenerate. Then every nonzero abelian inner ideal B of finite length of L is complemented by an abelian inner ideal. In fact, there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$ such that $B = L_n$. □

Proof. If L is nondegenerate and B is an abelian inner ideal of finite length, the subquotient $V = (B, L/\mathrm{Ker}_L\, B)$ is nondegenerate and Artinian by Proposition 3.5 (iii)(v). By [[18]; Theorem 5.2], V is covered by a finite grid, hence there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$ such that $B = L_n$ and B is complemented by the abelian inner ideal L_{-n}, see Theorem 6.1. ■

Theorem 6.3. Let $\mathcal{E}$ be a grid in a Jordan pair V with associated 3-graded root system (R, R_1). We enumerate $\mathcal{E} = \{e_\alpha : \alpha \in R_1\}$. For $\omega \in \mathbb{Z}^{R_1}$ we define $V_\omega(\mathcal{E}) = \big(V^+_\omega(\mathcal{E}), V^-_\omega(\mathcal{E})\big)$ by

$$V^+_\omega(\mathcal{E}) = \textstyle\bigcap_{\alpha\in R_1} V^+_{\omega(\alpha)}(g_\alpha) \quad \text{and} \quad V^-_\omega(\mathcal{E}) = \textstyle\bigcap_{\alpha\in R_1} V^-_{-\omega(\alpha)}(g_\alpha). \tag{6.2}$$

(a) Every $\omega \in \mathrm{supp}\, V = \{\omega \in \mathbb{Z}^{R_1} : V_\omega(\mathcal{E}) \neq 0\}$ has a unique extension to a weight of R, also denoted ω, such that $\omega(\alpha) = \langle \omega, \alpha^\vee \rangle$ holds for all $\alpha \in R_1$.

(b) Assume $V = \bigoplus_\omega V_\omega(\mathcal{E})$, which always holds if $\mathcal{E}$ is finite. Then, putting $V_\omega = 0$ for $\omega \in \mathcal{P}(R) \setminus \mathrm{supp}\, V$, the decomposition $V = \bigoplus_{\omega\in\mathcal{P}(R)} V_\omega(\mathcal{E})$ is a $\mathcal{P}(R)$-grading of V.

(c) Suppose $\mathcal{E}$ is finite. Then there exists a finite $\mathbb{Z}$-grading of V, say $V = \bigoplus_{i=-n}^{n} V_n$, satisfying

$$V^+ = \textstyle\bigoplus_{i=0}^{n} V^+_i, \quad V^- = \bigoplus_{i=-n}^{0} V^-_i, \quad \text{and} \quad e^\sigma_\alpha \in V^\sigma_{\sigma n} \text{ for all } \alpha \in R_1. \tag{6.3}$$

If $\mathcal{E}$ is connected, n can be taken as the Coxeter number of R. □

Proof. (a) and (b) can be proven in the same way as the proof of Theorem 5.5 (a) in 5.3, i.e., one verifies the conditions (i) and (ii) of Proposition 4.1 (b), see [20] for details. As in the proof of 6.1, the $\mathbb{Z}$-grading in (c) is then constructed from the $\mathcal{P}(R)$-grading using the homomorphism $\varphi\colon \mathcal{P}(R) \to \mathbb{Z}$ of Proposition 4.3. The properties mentioned in (6.3) are immediate from the definition (6.2). ■

Corollary 6.4. Let V be a Jordan pair, and let $B \subset V^+$ be an inner ideal of V whose subquotient is covered by a finite grid $\mathcal{G}$.

Then $\mathcal{G}$ lifts to a finite grid $\mathcal{E}$ in V such that the finite $\mathbb{Z}$-grading of V constructed in Theorem 6.3 satisfies $B = V_n^+$. Moreover, $C = V_{-n}^-$ is an inner ideal of V which complements B in the sense of [18]. □

Proof. Let L be the Tits-Kantor-Koecher algebra of V. We recall that $L = L^1 \oplus L^0 \oplus L^{-1}$ is a 3-graded Lie algebra with $L^{\pm 1} = V^{\pm}$. We will view V as a $\mathbb{Z}$-graded Jordan pair with respect to the grading induced from L, i.e., $V^1 = (V^+, 0)$ and $V^{-1} = (0, V^-)$.

By Proposition 3.3 every inner ideal of V contained in V^+ is an abelian inner ideal of L with $\mathrm{Ker}_L B = V^+ \oplus L^0 \oplus \mathrm{Ker}_V B$, whence B is a graded inner ideal of the 3-graded Lie algebra L whose subquotient $S = (B, L/\mathrm{Ker}_L B) \cong (B, V^-/\mathrm{Ker}_V B)$ is covered by a finite grid of (obviously) homogeneous idempotents. By repeated application of Proposition 5.4, the grid $\mathcal{G}$ lifts to a finite Peirce-compatible family $\mathcal{E}$ of idempotents of $\mathcal{L}$ which are homogeneous with respect to the 3-grading of L, whence $e^\sigma \in V^\sigma$ for $\sigma = \pm$ and $e = (e^+, e^-) \in \mathcal{E}$. By Lemma 5.6, $\mathcal{E}$ is a grid in V. We can then apply Theorem 6.3 and in particular get $B = \bigoplus_{\alpha \in R_1} V_\alpha^+(\mathcal{E}) = V_n^+$, $V_{-n}^- = \bigoplus_{\alpha \in R_1} V_{-\alpha}^-(\mathcal{E})$ and $\mathrm{Ker}_V B = \bigoplus_{\omega \notin R_{-1}} V_\omega^-(\mathcal{E}) = V_{-n+1}^- \oplus \cdots \oplus V_0^-$. It is obvious that $C = V_{-n}^-$ is an inner ideal of V. Applying what we just proved to C and V^{op} shows $\mathrm{Ker}_V C = V_0^+ \oplus \cdots V_{n-1}^+$. ■

6.1 Abelian Inner Ideals in Simple Finite-Dimensional Lie Algebras

By Corollary 6.2, a description of abelian inner ideals in nondegenerate Artinian Lie algebras can be deduced from a classification of finite $\mathbb{Z}$-gradings of these Lie algebras. Although this is not very efficient since nonisomorphic $\mathbb{Z}$-gradings can lead to isomorphic abelian inner ideals, it nevertheless provides a quick classification of abelian inner ideals of those Lie algebras for which the finite $\mathbb{Z}$-gradings are known.

As an example, we consider in this subsection a finite-dimensional split simple Lie algebra L over a field Φ of characteristic 0, and let $B \subset L$ be an abelian inner ideal. By Corollary 6.2, L has a finite $\mathbb{Z}$-grading, say a $(2n+1)$-grading, with $L_n = B$. It is folklore that the $\mathbb{Z}$-gradings of L are obtained as follows: There exists a splitting Cartan subalgebra $\mathfrak{h}$ of L and a $\mathbb{Z}$-grading of the root system R of $(L, \mathfrak{h})$, say $R = \bigcup_{i=-n}^n R_n$ such that $L_i = \bigoplus_{\alpha \in R_i} L_\alpha$, where the L_α are the root spaces of $(L, \mathfrak{h})$, in particular $L_n = B = \sum_{\alpha \in R_n} L_\alpha$. It is therefore enough to determine R_n. This can be done as follows, see e.g. [[19]; 17.4, 17.5]: Any $\mathbb{Z}$-grading $(R_i)_{i \in \mathbb{Z}}$ of R is given by a coweight q of R, i.e. a $\mathbb{Z}$-linear map $\mathcal{Q}(R) \to \mathbb{Z}$, via $R_i = \{\alpha \in R : q(\alpha) = i\}$, and any coweight q is uniquely determined by its values $q_i = q(\beta_i)$, where $(\beta_1, \ldots, \beta_l)$ is a root basis of R. One then discusses the possibilities for the family (q_i) keeping in mind that the highest root with respect to $(\beta_1, \ldots, \beta_l)$ lies in

R_n. As an example, we will give the classification for $R = \mathrm{E}_8$ below. Before doing so, we mention some general facts for L:

- By [[4]; Lemma 1.13] every proper inner ideal of a simple nondegenerate Artinian Lie algebra is abelian, hence in particular this is so for proper inner ideals of L.
- The inner ideals coming from a 3-grading of L are well known: They are the V^+-spaces of simple Jordan pairs V whose Tits-Kantor-Koecher algebra is (isomorphic to) L. Moreover, by Lemma 2.3 (a), a submodule $B \subset V^+$ is an inner ideal of the Lie algebra L if and only if B is an inner ideals of the Jordan pair V. The latter are well known, see e.g. [21] or [[24]; Section 3].
- If $\mathcal{B} \subset R$ is a family of pairwise collinear long roots, then $B = \bigoplus_{\beta\in\mathcal{B}} L_\beta$ is an abelian inner ideal. This is easily proven using standard facts from root systems. We note that with $b = |\mathcal{B}|$ the corresponding subquotient is isomorphic to the rectangular matrix pair $\big(\mathrm{Mat}(1, b, \Phi), \mathrm{Mat}(b, 1; \Phi)\big) =$ (I_{1b} in the notation of [16]), and hence the subalgebra $\mathfrak{g}$ of Theorem 5.5 is isomorphic to $\mathfrak{sl}_{b+1}(\Phi)$.

Example $R = \mathrm{E}_8$**:** We will use the enumeration of the simple roots β_i as in [[6]; Planche VII], and let B be the abelian inner ideal associated to R_n with n as in (6.1). To arrive at the following list of isomorphism classes of abelian inner ideals in E_8 one considers the possibilities for q_i, $1 \le i \le 8$, starting with q_8. If $q_8 > 0$, then in view of the known coefficients (m_i) of any positive root $\alpha = \sum_{i=1}^{8} m_i\beta_i$ (see [[6]; Planche VII]) we have $|R_n| = 1$. If $q_8 = 0 < q_7$ then $|R_n| = 2$. Continuing in this way one arrives at the following list:

(1) R_n is a family of pairwise collinear roots, $1 \le |R_n| \le 8$. For example, $|R_n| = 8$ is obtained from $q_2 > 0 = q_3 = \cdots = q_8$.

(2) R_n is a family of 14 roots, obtained from $q_1 > 0 = q_2 = \cdots = q_8$. Two distinct roots in R_n are either orthogonal or collinear, and there exists a bijection between R_n and the idempotents in an even quadratic form grid of 14 idempotents, that preserves orthogonality and collinearity. The corresponding subquotient is therefore a quadratic form pair of dimension 14 (IV_{14} in the notation of [16]), and the Lie algebra $\mathfrak{g}$ of Theorem 5.5 is of type D_8. The corresponding grading of L is a 5-grading. R_i consists of those roots whose β_1-coefficient is i; we have $|R_1| = 64$ and $|R_0| = 84$. (L_1, L_{-1}) is the Kantor pair of the structurable algebra $\mathbb{O} \otimes \mathbb{O}$, where $\mathbb{O}$ is a split octonion.

6.2 Abelian Inner Ideals of Finite Length in Simple Infinite-Dimensional Lie Algebras.

In the previous subsection we have seen the relationship between abelian inner ideals (of finite length) and finite $\mathbb{Z}$-gradings in a finite-dimensional split simple Lie algebra L over a field Φ of characteristic 0. Let us now analyze this relation in the case of an infinite-dimensional simple Lie algebra L over a field Φ of characteristic 0. Let B be a nontrivial abelian inner ideal of finite length of L. Then L has abelian minimal inner ideals, so (see [7]) $L = \mathrm{Soc}(L)$ is 5-graded and there exists a simple associative algebra A with nonzero socle such that

(i) $L = [A,A]/([A,A] \cap Z(A))$ with the induced product of $A^{(-)}$, or

(ii) $L = [K,K]/Z(A) \cap [K,K]$, where $*$ is an involution of A, $K = \mathrm{Skew}(A,*)$ and either $Z(A) = 0$ or the dimension of A over $Z(A)$ is greater than 16.

On the other hand, by Corollary 6.2, L has a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$ for which $B = L_n$. Let us now see that Corollary 6.2 indeed holds for $n = 2$, i.e., there exists a 5-grading of L for which $L_2 = B$.

Proposition 6.5. Let A be an associative algebra and let $L = [A,A]/([A,A] \cap Z(A))$.

(a) Let $e,f \in A$ be idempotents satisfying $fe = 0$. Then $B = eAf$ is an abelian inner ideal of $A^{(-)}$, which is contained in $[A,A]$ and which satisfies $B \cap Z(A) = \{0\}$. Hence B imbeds into the Lie algebra $L = [A,A]/([A,A] \cap Z(A))$ and is an abelian inner ideal in L. Moreover:

(i) $c = f - ef$ is an idempotent of A which is orthogonal to e and also satisfies $B = eAc$.

(ii) There exists a 5-grading of A as associative algebra, $A = \bigoplus_{i=-2}^{2} A_i$, such that

$$B = A_2 \quad \text{and}$$

$$\mathrm{Ker}_{A^{(-)}} B = \{x \in A_{-2} : bxb = 0 \text{ for all } b \in B\} \oplus \bigoplus_{i \geq -1} A_i.$$

Assume that A is semiprime or that there exist $u_2 \in A_2$ and $v_{-2} \in A_{-2}$ satisfying $u_2 v_{-2} = e$ and $v_{-2} u_2 = c$. Then $\mathrm{Ker}_{A^{(-)}} B = \bigoplus_{i \geq -1} A_i$ and the subquotient of B is isomorphic to the Jordan pair $V = (A_2, A_{-2})$ with triple product $\{a,b,c\} = abc + cba$.

(b) Conversely, if A is simple then every abelian inner ideal $B \subset L$ of finite length is of the form $B = eAf$, where e,f are orthogonal idempotents. Moreover, the Jordan pair $V = (A_2, A_{-2})$ described in (a) is simple and Artinian. $\square$

Proof.

(a) Clearly $B^2 = 0$, and this easily implies that $B = eAf$ is an inner ideal of $A^{(-)}$. It is also straightforward to check that c is an idempotent of A which is orthogonal to e and satisfies $ef = efc$. From this one deduces that $B = eAc$ and then that $eac = [e, eac]$ for any $a \in A$. Hence $B \subset [A, A]$ and $B \cap Z(A) = 0$. Let $\widehat{A} = \Phi 1 \oplus A$ be the associative algebra obtained from A by adjoining a unit element 1. Then $(e_1, e_2, e_3) = (e, 1 - c - e, c)$ is a complete orthogonal system in $\widehat{A}$. Let $\widehat{A}_{jk}$ be the corresponding Peirce spaces, hence $\widehat{A} = \bigoplus_{1 \le i,j \le 3} \widehat{A}_{jk}$. Since $A\widehat{A} + \widehat{A}A \subset A$ all Peirce spaces $\widehat{A}_{jk}$ with $(jk) \neq (22)$ are in fact contained in A and can be defined in A, e.g. $\widehat{A}_{11} = \{a \in A : ea = a = ae\}$, $\widehat{A}_{12} = \{a \in A : ea = a, 0 = a(e + c)\}$. For $(jk) = (22)$ we have $\widehat{A}_{22} = \Phi e_2 \oplus A_{22}$ where $A_{22} = \{a \in A : (e + c)a = 0 = a(e + c)\}$. We therefore get a decomposition $A = \bigoplus_{1 \le j,k \le 3} A_{jk}$ with $A_{jk} = \widehat{A}_{jk}$ for $(jk) \neq (22)$ which behaves like a Peirce decomposition. Put $A_i = \bigoplus_{k-j=i} A_{jk}$ for $-2 \le i \le 2$. Then $A = \bigoplus_{i=-2}^{2} A_i$ is a 5-grading of A with $B = eAc = A_{13} = A_2$. The remaining claims of (a) can now easily be checked.

(b) That in a simple Artinian associative algebra A every abelian inner ideal of L has the form eAf with $fe = 0$ is shown in [[4]; Theorem 5.1]. Let us then suppose that L is not Artinian and let B be a nonzero abelian inner ideal of finite length. By socle theory for Lie algebras [[7], Theorem 4.5], A has nonzero socle (as an associative algebra), and since it is not Artinian, $Z(A) = 0$. Therefore $L = [A, A]$. By [[3], Lemma 3.14], $b^2 = 0$ for any $b \in B$. Hence, for any $b, c \in B$ and $a \in A$, we have $[[b, a], c] = bac + cab \in A$, which implies that B is an inner ideal of the Jordan algebra $A^{(+)}$. But inner ideals of finite length of $A^{(+)}$ are of the form eAf with e, f idempotents of A [[8], (16)]. Since $b^2 = 0$ for any $b \in B$, we have $fe = 0$. Indeed, $b^2 = 0$ for any $b \in B$ implies $bc + cb = 0$ for any $b, c \in B$; on the other hand, $bc - cb = 0$ for any $b, c \in B$ since B is abelian, hence $B^2 = 0$. Then it follows by simplicity of A that $fe = 0$ (otherwise, $fe \neq 0$ would imply $A = AfeA$, and hence $B = eAf = eAfeAf = B^2 = 0$). ■

Remark 6.6. Recall [13] that a simple associative algebra A with an involution $*$ has nonzero socle if and only if it is $*$-isomorphic to the algebra of finite rank continuous operators $(\mathcal{F}(X), *)$, where X is a left vector space endowed with a nondegenerate skew-Hermitian or symmetric form h over a division algebra with involution $(\Delta, -)$, and where $*$ denotes the adjoint involution. In the last case, Δ is commutative with the identity as involution and $K = \mathrm{Skew}(A, *)$ is the finitary orthogonal algebra $\mathfrak{fo}(X, h)$ [1]. Given $x, y \in X$, we write x^*y to denote the linear operator defined by $x^*y(x') = h(x', x)y$ for all $x' \in X$. Then $x^*y \in \mathcal{F}(X)$ with $(x^*y)^* = y^*x$. Hence $[x, y] := x^*y - y^*x \in \mathfrak{fo}(X, h)$.

Proposition 6.7. Let A be a simple associative algebra with involution $*$ such that either $Z(A) = 0$ or the dimension of A over $Z(A)$ is greater than 16. Put $K = \mathrm{Skew}(A, *)$ and $L = [K, K]/Z(A) \cap [K, K]$.

(a) If B is an abelian inner ideal of L of finite length, then either

(i) $B = eKe^*$ for $e \in A$ an idempotent such that e and e^* are orthogonal, or

(ii) $L = \mathfrak{fo}(X, h)$ as in Remark 6.6 and there exist a hyperbolic plane $H \subset X$ and a nonzero isotropic vector $x \in H$ such that $H^\perp$ does not contain infinite dimensional totally isotropic subspaces and B is given by $B = [x, H^\perp] := \{[x, z] : z \in H^\perp\}$.

(b) If $B = eKe^*$ as in (i), then A has a 5-grading as an associative algebra, $A = A_{-2} \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus A_2$, which is induced by the idempotents e and e^*, cf. Proposition 6.5. Moreover, L is 5-graded with $B = eKe^* = L_2$.

If $B = [x, H^\perp] \subset L = \mathfrak{fo}(X, h)$ as in (ii), then L admits a 3-grading, $L = L_{-1} \oplus L_0 \oplus L_1$, such that $B = [x, H^\perp] = L_1$. □

Proof.

(a) We may assume $B \neq 0$, and therefore that L has nonzero socle. Then, by [[7]; Theorem 5.16], A coincides with its socle. We consider two cases.

(1) $b^2 = 0$ for any $b \in B$. If A is Artinian, then we have by [[3]; Theorem 5.5] that $B = eKe^*$ for some idempotent e in A satisfying $e^*e = 0$. As in Proposition 6.5 we may assume that the idempotents e and e^* are orthogonal. If A is not Artinian, then $Z(A) = 0$ and $L = [K, K]$. It then follows from [[9]; Prop. 3.6] that $B = eKe^*$ as before.

(2) $b^2 \neq 0$ for some $b \in B$. Then we have by [[9]; Proposition 3.8] that Δ is a field with the identity as involution and $B = [x, H^\perp]$. Moreover, by [[9]; Lemma 3.7], $H^\perp$ cannot contain infinite dimensional totally isotropic subspaces.

(b) The case $B = eKe^*$ follows as in the proof of Proposition 6.5 (note that $A_i^* = A_i$). If $B = [x, H^\perp] = L_1$ as in (ii), let e_0, e_1, e_2 be the canonical projections of $X = Fx_+ \oplus H^\perp \oplus Fx_-$ onto Fx_+, $H^\perp$, Fx_-, respectively. It is easy to see that e_0, e_1, e_2 are idempotents in $\mathcal{L}(X)$, the algebra of all continuous operators, with $e_0^* = e_2$ and $e_1^* = e_1$, which induce a 5-grading $A = A_2 \oplus A_1 \oplus A_0 \oplus A_{-1} \oplus A_{-2}$ of the simple associative algebra $A = \mathcal{F}(X)$. Moreover, each A_i is invariant under $*$ and $\mathrm{Skew}(A_2, *) = \mathrm{Skew}(A_{-2}, *) = 0$. Hence $\mathfrak{fo}(X, q) = \mathrm{Skew}(\mathcal{F}(X), *) = L_1 \oplus L_0 \oplus L_{-1}$ with $L_i = \mathrm{Skew}(A_i, *)$ and $B = L_1$. ∎

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